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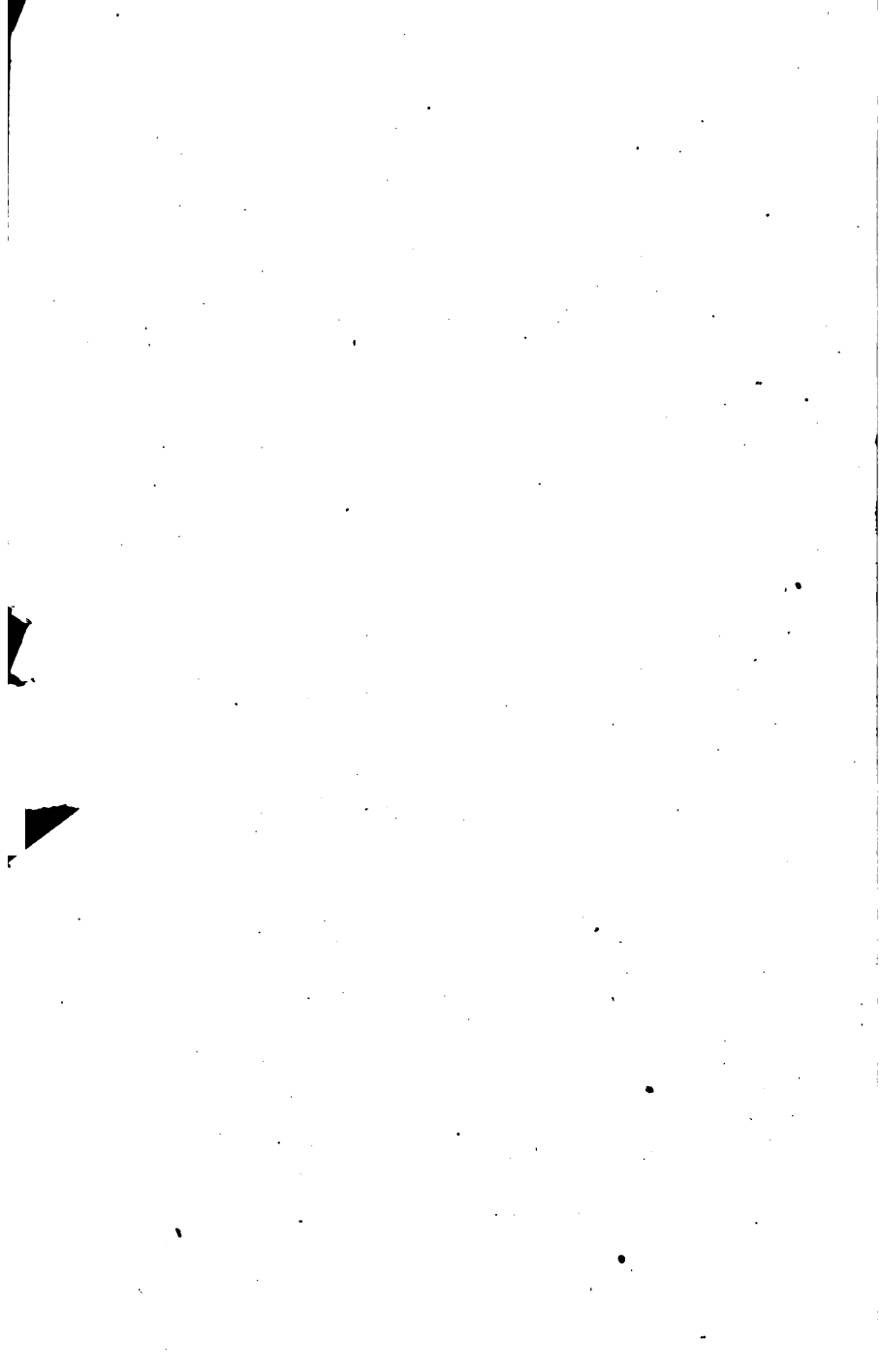
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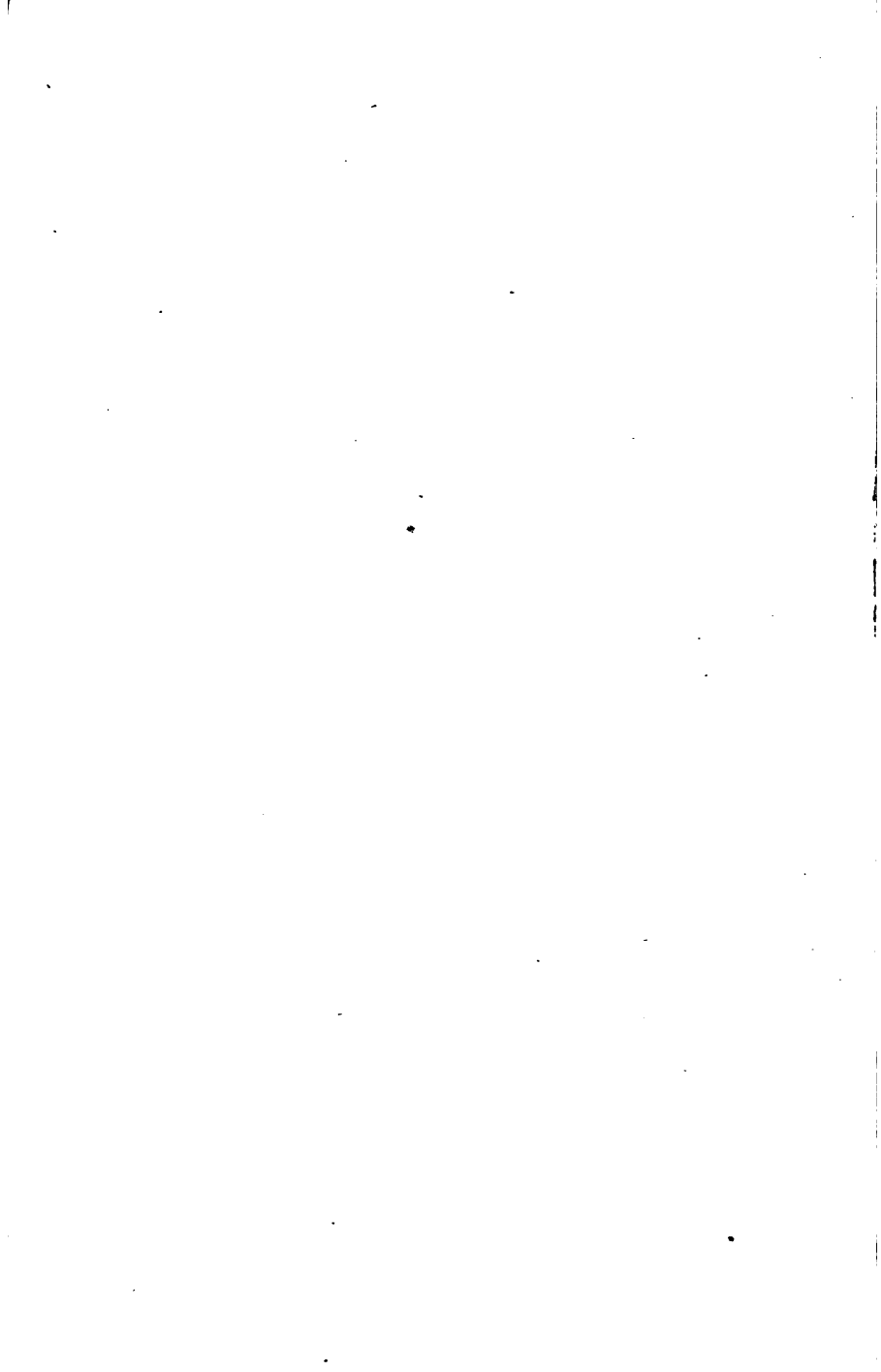
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KEY

TO THE

ELEMENTARY ALGEBRA.

BY BENJAMIN GREENLEAF, A. M.,

AUTHOR OF A MATHEMATICAL SERIES.

FOR THE USE OF TEACHERS ONLY.

BOSTON:
PUBLISHED BY ROBERT S. DAVIS & CO.

NEW YORK: WILLIAM WOOD AND COMPANY, AND W. I. POOLEY.

PHILADELPHIA: J. B. LIPPINCOTT AND COMPANY.

ST. LOUIS: KEITH AND WOODS.

NEW ORLEANS: DUNCAN & CO.

1866.

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KEY

TO

GREENLEAF'S NEW ELEMENTARY

ALGEBRA.

DEFINITIONS AND NOTATION.

SIGNS.

EXAMPLES, page 9.

- | | |
|--|--|
| <p>2. Ans. 14.</p> <p>4. Ans. 10.</p> <p>7. Ans. $44 - 5 = 39$.</p> <p>9. Ans. $120 \div 12 = 10$.</p> | <p>11. Ans. $108 - 6 = 102$.</p> <p>12. Ans. $16 > 12$.</p> <p>13. Ans. $4 < 6$.</p> |
|--|--|

ALGEBRAIC EXPRESSIONS.

EXAMPLES, pp. 15, 16.

- | | |
|---|--|
| <p>4. Ans. $x + 2y - z$.</p> <p>8. Ans. $\frac{4a}{3c}$.</p> <p>9. Ans. $\frac{a-b}{ab}$.</p> <p>11. Ans. $2a + \frac{b}{c}$.</p> | <p>13. Ans. $\frac{a^4 - b^5}{a - b^2}$.</p> <p>15. Ans. $\frac{15a^2 + b^5}{a^2 - b^2} + 2c$.</p> <p>17. Ans. $\frac{1}{ab^2} - \frac{1}{a^2 + c^2}$.</p> <p>21. Ans. $\sqrt[3]{x} - \sqrt{x}$.</p> |
|---|--|

INTERPRETATION OF ALGEBRAIC EXPRESSIONS

EXAMPLES, pp. 17, 18.

1. $12 + 3 - 2 + 4 = 17.$
2. $36 + 2 - 4 = 34.$
3. $48 - 15 + 8 - 28 = 13.$ Ans. 13.
4. $(12 - 3) \times (2 + 4) = 9 \times 6 = 54.$
5. $81 \times 2 + 4 = 162 + 4 = 166.$
6. $\frac{15}{5} + 45 = 3 + 45 = 48.$
7. $4 \times 15 - \frac{12}{4} = 60 - 3 = 57.$ Ans. 57.
8. $4 \times 9 - 5 \times 4 = 36 - 20 = 16.$
9. $2 \times 144 \times 2 - \frac{144}{2} + \frac{12}{4} = 576 - 72 + 3 = 507.$
10. $\frac{12 + 9 + 8}{13} \times \frac{144 - 27}{9} = \frac{29}{13} \times 13 = 29.$
11. $\left(\frac{11}{11} + 12\right) \times (3 - 2) - 4 = 13 \times 1 - 4 = 9.$
12. $256 - 256 + 12 - 6 = 6.$ Ans. 6.
13. $252 + 1 \times 1 = 253.$
14. $60 - 7 \times 4 = 60 - 28 = 32.$
15. $4 \times 4 + 8 \times 12 - 112 = 16 + 96 - 112 = 0.$ Ans. 0.
16. $\sqrt{16} + \sqrt{100} - \sqrt{81} = 4 + 10 - 9 = 5.$
17. $12\sqrt{25 - 24} + \sqrt{48 + 16} = 12 + 8 = 20.$
18. $\sqrt{81} - \sqrt[3]{8} + \sqrt{4} = 9 - 2 + 2 = 9.$
19. $10 + 8\sqrt{16} - 2\sqrt[3]{8} = 10 + 32 - 4 = 38.$

ALGEBRAIC PROCESSES.

(ART. 46, pp. 20-22.)

2. Let x = number of cents spent,
and $2x$ = number of cents left.
Then $3x = 45$ cents, the whole number,
and $x = 15$ cents, the number spent.
3. Let x = number of apples each has.
Then $2x = 56$ apples, the whole number,
and $x = 28$ apples, the number each has.
Ans. 28 apples.
4. Let x = length of part left standing,
and $3x$ = length of part broken off.
Then $4x = 60$ feet, the whole length,
 $x = 15$ feet, the part left standing,
and $3x = 45$ feet, the part broken off.
5. Let x = the less number,
and $5x$ = the greater number.
Then $6x = 126$, their sum;
 $x = 21$, the less number,
and $5x = 105$, the greater number.
6. Let x = value of the chaise,
and $3x$ = value of the horse.
Then $4x = 340$ dollars, the value of both,
 $x = 85$ dollars, the value of the chaise,
and $3x = 255$ dollars, the value of the horse.
7. Let x = A's share,
and $4x$ = B's share.
Then $5x = 2500$ dollars, the whole sum,
 $x = 500$ dollars, A's share,
and $4x = 2000$ dollars, B's share.

9. Let $x = A$'s share,
 $2x = B$'s share,
and $2x = C$'s share.
Then $5x = 300$ dollars, the whole sum,
 $x = 60$ dollars, A 's share,
 $2x = 120$ dollars, B 's share,
and $2x = 120$ dollars, C 's share.
10. Let $x =$ price of the apples,
 $2x =$ price of the pears,
and $4x =$ price of the oranges.
Then $7x = 63$ cents, the sum paid for the whole,
 $x = 9$ cents, the price of the apples,
 $2x = 18$ cents, the price of the pears,
and $4x = 36$ cents, the price of the oranges.
Ans. Apples, 9 cts.; pears, 18 cts.; and oranges, 36 cts.
11. Let $x = A$'s age,
 $2x = B$'s age,
and $3x = C$'s age.
Then $6x = 78$ years, the sum of their ages,
 $x = 13$ years, A 's age,
 $2x = 26$ years, B 's age,
and $3x = 39$ years, C 's age.
12. Let $x =$ price of the sheep,
 $7x =$ price of the cow,
and $28x =$ price of the horse.
Then $36x = 180$ dollars, the amount received for all,
 $x = 5$ dollars, the price of the sheep,
 $7x = 35$ dollars, the price of the cow,
and $28x = 140$ dollars, the price of the horse.
13. Let $x =$ the first number,
 $4x =$ the second number,
and $2x =$ the third number.

- Then $7x = 350$, their sum,
 $x = 50$, the first number,
 $4x = 200$, the second number,
 and $2x = 100$, the third number.
14. Let $x =$ distance traveled the first day,
 $3x =$ distance traveled the second day,
 and $2x =$ distance traveled the third day.
 Then $6x = 84$ miles, the whole distance,
 $x = 14$ miles, distance traveled the first day,
 $3x = 42$ miles, dist. traveled the second day,
 and $2x = 28$ miles, dist. traveled the third day.
15. Let $x =$ sum contributed by A,
 $4x =$ sum contributed by B,
 and $3x =$ sum contributed by C.
 Then $8x = 600$ dollars, the whole sum,
 $x = 75$ dollars, sum contributed by A,
 $4x = 300$ dollars, sum contributed by B,
 and $3x = 225$ dollars, sum contributed by C.

ADDITION.

(ART. 49, pp. 23-25.)

7. Ans. — $21 \delta x$. | 9. Ans. $17 a + 5 b$.
 8. Ans. $17 m n^2$. | 10. Ans. $12 c^2 d - 5 a^3 c$.

$$\begin{array}{r}
 (13.) \\
 2x + 3y \\
 x + 8y \\
 3x + y \\
 6x + 2y \\
 x + 4y \\
 4x + y \\
 \hline
 17x + 19y
 \end{array}$$

$$\begin{array}{r}
 (14.) \\
 7a^2 - b \\
 3a^2 - 3b \\
 6a^2 - 2b \\
 2a^2 - b \\
 4a^2 - 6b \\
 a^2 - 4b \\
 \hline
 23a^2 - 17b
 \end{array}$$

(ART. 50, pp. 25-27.)

4. Ans. $24 ax$.

6. Ans. $x^2 - 5xy^2$.

$$\begin{array}{r}
 (7.) \\
 a - b \\
 - 2a - 7b \\
 7a - 2b \\
 - 3a + 3b \\
 - 8a + b \\
 a + 7b \\
 \hline
 - 4a + b
 \end{array}$$

$$\begin{array}{r}
 (8.) \\
 5cd^2 + 7a^2b \\
 - 3cd^2 - 5a^2b \\
 9cd^2 - 10a^2b \\
 - 4cd^2 + a^2b \\
 \hline
 7cd^2 - 7a^2b
 \end{array}$$

(ART. 51, pp. 27, 28.)

$$\begin{array}{r}
 (5.) \\
 -a^2b - cd^2 \\
 -a^2b + cd^2 \\
 + cd^2 - 3ab \\
 5a^2b + cd^2 \\
 \hline
 \text{Ans. } 3a^2b + 2cd^2 - 3ab
 \end{array}$$

$$\begin{array}{r}
 (6.) \\
 3x - 7y + 2z \\
 -x + 4y + 6z \\
 -2y - 3z + a \\
 4x - y + 3z \\
 \hline
 6x - 6y + 8z + a
 \end{array}$$

$$\begin{array}{r}
 (7.) \\
 -5ax + 2by - 7 \\
 3by + 18 - 4x \\
 4ax - by - 9 \\
 3ax - 2by + 26 \\
 \hline
 2ax + 2by + 28 - 4x
 \end{array}$$

$$\begin{array}{r}
 (8.) \\
 x^3 + ax^2 + bx + 2 \\
 3x^3 - 4ax^2 + 7 - 6bxy \\
 3x^3 - 3ax^2 - 7bx - 19 \\
 \hline
 7x^3 - 6ax^2 - 6bx - 10 - 6bxy
 \end{array}$$

$$\begin{array}{r}
 (9.) \\
 8a^2x^2 - 3ax \\
 7ax - 5xy \\
 - 5ax + 9xy - b^2c^2 \\
 2a^2x^2 + xy \\
 \hline
 10a^2x^2 - ax + 5xy - b^2c^2
 \end{array}$$

(ART. 52, pp. 28, 29.)

(2.)	(3.)	(4.)
$5(a+x)$	$3(x^2-a)$	$4\sqrt{a-x}$
$6(a+x)$	$2(x^2-a)$	$3\sqrt{a-x}$
$8(a+x)$	$-(x^2-a)$	$-7\sqrt{a-x}$
$3(a+x)$	$6(x^2-a)$	$2\sqrt{a-x}$
$(a+x)$	(x^2-a)	$\sqrt{a-x}$
$23(a+x)$	$11(x^2-a)$	$3\sqrt{a-x}$

(5.)	(6.)
$7y-4(a+b)$	$2(x-y)^2$
$6y+2(a+b)$	$3(x-y)^2$
$2y+(a+b)$	$(x-y)^2$
$y-3(a+b)$	$-(x-y)^2 + (x+y)$
$16y-4(a+b)$	$(x+y)$
	$5(x-y)^2 + 2(x+y)$

(ART. 53, pp. 29, 30.)

(2.)	(3.)	(4.)
bx	$3ay$	$(a+b)x$
abx	$-cy$	$(a-c)x$
$2cx$	$-2ay$	$(2a+b-c)x$
$(b+ab+2c)x$	$(a-c)y$	

(5.)	(6.)
$(a+b)x$	$ax+b$
$2cx$	$cx+d$
$2x$	$(a+c)x+b+d$
$(a+b+2c+2)x$	

(7.)

$$\begin{array}{r}
 ax + 7m \\
 7ax - 3m \\
 bx + 4m \\
 \hline
 (8a + b)x + 8m
 \end{array}$$

(8.)

$$\begin{array}{r}
 ax^2 + bx \\
 cx^2 - dx \\
 \hline
 (a + c)x^2 + (b - d)x
 \end{array}$$

SUBTRACTION.

(ART. 54, pp. 30-34.)

10. Ans. 11 y . 12. Ans. 3 ax^2 . 19. Ans. 0.
 11. Ans. 9 a . 13. Ans. $-16x^2y^3$. 21. Ans. $-15d$.

(22.)

$$\begin{array}{r}
 a + b \\
 -a \\
 \hline
 2a + b
 \end{array}$$

(23.)

$$\begin{array}{r}
 a + b \\
 a - b \\
 \hline
 2b
 \end{array}$$

(24.)

$$\begin{array}{r}
 a - b \\
 a + b \\
 \hline
 -2b
 \end{array}$$

(25.)

$$\begin{array}{r}
 a - b \\
 -a + b \\
 \hline
 2a - 2b
 \end{array}$$

(26.)

$$\begin{array}{r}
 5xy \\
 3xy - 3 \\
 \hline
 2xy + 3
 \end{array}$$

(27.)

$$\begin{array}{r}
 a + b + c \\
 a - b - c \\
 \hline
 2b + 2c
 \end{array}$$

(28.)

$$\begin{array}{r}
 x \\
 x + y \\
 \hline
 -y
 \end{array}$$

(29.)

$$\begin{array}{r}
 x + 5 \\
 y - 2 \\
 \hline
 x - y + 7
 \end{array}$$

(32.)

$$\begin{array}{r}
 2x^2 - y^2 \\
 -2x^2 - y^2 \\
 \hline
 4x^2
 \end{array}$$

(35.)

$$\begin{array}{r}
 5xy + 2x^2 + 2a \\
 3xy - x^2 - 7a \\
 \hline
 2xy + 3x^2 + 9a
 \end{array}$$

(36.)

$$\begin{array}{r}
 6abx + 12 - 3xy \\
 3abx - 7 \\
 \hline
 3abx + 19 - 3xy
 \end{array}$$

(37.)

$$\begin{array}{r}
 ax^2 - bx^2 + cx \\
 + bx^2 + cx - 12d \\
 \hline
 ax^2 - 2bx^2 + 12d
 \end{array}$$

(38.)

$$\begin{array}{r}
 3a + b + c - d \\
 3a + b - 18 \\
 \hline
 c - d + 18
 \end{array}$$

(39.)

$$\begin{array}{r}
 5x \quad \quad - b \\
 - 2xy + b \\
 \hline
 5x + 2xy - 2b
 \end{array}$$

(40.)

$$\begin{array}{r}
 3a(a-y) + 4by + a^3 \\
 2a(a-y) - 7by + 4a^3 \\
 \hline
 a(a-y) + 11by - 3a^3
 \end{array}$$

(41.)

$$\begin{array}{r}
 a^3 + 3b^2c + ab^2 - abc \\
 + ab^2 - abc + b^3 \\
 \hline
 a^3 + 3b^2c \quad \quad - b^3
 \end{array}$$

(42.)

$$\begin{array}{r}
 - 6a - 4b - 12c + 12x \\
 8a + 4b - 5c - 2x \\
 \hline
 - 14a - 8b - 7c + 14x
 \end{array}$$

(43.)

$$\begin{array}{r}
 2ab + b^2 - 4c + bc \\
 3ab + 2b^2 - c - 3bc + 4b^3 \\
 \hline
 -ab - b^2 - 3c + 4bc - 4b^3 = -ab - 3c + 4bc - 5b^3
 \end{array}$$

(ART. 55, pp. 34, 35.)

$$6. \quad -6a^2 - b - 4a^2 - b^2 = -10a^2 - b - b^2.$$

$$9. \quad \text{Ans. } 14a^2 - b^2 + c^2.$$

(ART. 56, p. 36.)

(4.)

$$\begin{array}{r}
 ay + by \\
 cy + by \\
 \hline
 (a-c)y
 \end{array}$$

(5.)

$$\begin{array}{r}
 ax - b \\
 cx \quad - d \\
 \hline
 (a-c)x - b + d
 \end{array}$$

(6.)

$$\begin{array}{r}
 ay - by + cy \\
 ay - by + y \\
 \hline
 (c-1)y
 \end{array}$$

(7.)

$$\begin{array}{r}
 ax^2 \quad + bx \\
 cx^2 \quad - cx \\
 \hline
 (a-c)x^2 + (b+c)x
 \end{array}$$

(8.)

$$\begin{array}{r}
 a x^2 \qquad + m x y \qquad + n x + b \\
 c x^2 \qquad - d x y \qquad + e x \qquad - z \\
 \hline
 (a - c) x^2 + (m + d) x y + (n - e) x + b + z
 \end{array}$$

MULTIPLICATION.

(ART. 62, pp. 38-40.)

$$\begin{array}{lcl}
 7. & \text{Ans. } 4 x^2. & 9. \qquad \text{Ans. } 12 a b. \\
 8. & \text{Ans. } -34 a b. & 10. \qquad \text{Ans. } -28 b c.
 \end{array}$$

(ART. 63, pp. 41, 42.)

$$\begin{array}{l}
 5. \text{ Ans. } 7 x^3 + 4 x^2 y + a^2 x^2. \\
 6. \text{ Ans. } -20 a^2 b x^2 + 4 a^3 x^3 - 4 a x^3.
 \end{array}$$

(ART. 64, pp. 42-44.)

$$\begin{array}{rcl}
 (4.) & & (5.) \\
 \begin{array}{r}
 3 a^2 - 2 y \\
 x + y \\
 \hline
 3 a^2 x - 2 x y \\
 \qquad + 3 a^2 y - 2 y^2 \\
 \hline
 \text{Ans. } 3 a^2 x - 2 x y + 3 a^2 y - 2 y^2
 \end{array} & & \begin{array}{r}
 a - b \\
 a - b \\
 \hline
 a^2 - a b \\
 \qquad - a b + b^2 \\
 \hline
 \text{Ans. } a^2 - 2 a b + b^2
 \end{array}
 \end{array}$$

$$\begin{array}{rcl}
 (6.) & & (9.) \\
 \begin{array}{r}
 5 a x + 3 x \\
 3 a x + 2 x \\
 \hline
 15 a^2 x^2 + 9 a x^2 \\
 \qquad 10 a x^2 + 6 x^2 \\
 \hline
 \text{Ans. } 15 a^2 x^2 + 19 a x^2 + 6 x^2
 \end{array} & & \begin{array}{r}
 3 x + 2 y \\
 2 x - 3 y \\
 \hline
 6 x^2 + 4 x y \\
 \qquad - 9 x y - 6 y^2 \\
 \hline
 6 x^2 - 5 x y - 6 y^2
 \end{array}
 \end{array}$$

(10.)

$$\begin{array}{r} 5a^2 + 3x \\ 5a^2 + 3x \\ \hline 25a^4 + 15a^2x \\ \quad + 15a^2x + 9x^2 \\ \hline 25a^4 + 30a^2x + 9x^2 \end{array}$$

(11.)

$$\begin{array}{r} a + 2x \\ a - 3x \\ \hline a^2 + 2ax \\ \quad - 3ax - 6x^2 \\ \hline a^2 - ax - 6x^2 \end{array}$$

(12.)

$$\begin{array}{r} 3a - x \\ 2a + 4x \\ \hline 6a^2 - 2ax \\ \quad + 12ax - 4x^2 \\ \hline 6a^2 + 10ax - 4x^2 \end{array}$$

(13.)

$$\begin{array}{r} x + y \\ x + y \\ \hline x^2 + xy \\ \quad xy + y^2 \\ \hline x^2 + 2xy + y^2 \end{array}$$

(14.)

$$\begin{array}{r} x - y \\ x + y \\ \hline x^2 - xy \\ \quad xy - y^2 \\ \hline x^2 - y^2 \end{array}$$

(15.)

$$\begin{array}{r} a^2 + ab + b^2 \\ a - b \\ \hline a^3 + a^2b + ab^2 \\ \quad - a^2b - ab^2 - b^3 \\ \hline a^3 - b^3 \end{array}$$

(16.)

$$\begin{array}{r} a^2 - a + 1 \\ a + 1 \\ \hline a^3 - a^2 + a \\ \quad a^2 - a + 1 \\ \hline a^3 + 1 \end{array}$$

(17.)

$$\begin{array}{r} x^3 - ax^2 + a^2x - a^3 \\ x + a \\ \hline x^4 - ax^3 + a^2x^2 - a^3x \\ \quad ax^3 - a^2x^2 + a^3x - a^4 \\ \hline x^4 - a^4 \end{array}$$

(18.)

$$\begin{array}{r} a^4 - a^3y + a^2y^2 - ay^3 + y^4 \\ a + y \\ \hline a^5 - a^4y + a^3y^2 - a^2y^3 + ay^4 \\ \quad a^4y - a^3y^2 + a^2y^3 - ay^4 + y^5 \\ \hline a^5 + y^5 \end{array}$$

(19.)

$$\begin{array}{r} x^2 + y \\ x^2 + y \\ \hline x^4 + x^2y \\ \quad x^2y + y^3 \\ \hline x^4 + 2x^2y + y^3 \end{array}$$

(20.)

$$\begin{array}{r}
 2a b - 3b^2 \\
 3a b + 4b^2 \\
 \hline
 6a^2 b^2 - 9a b^3 \\
 8a b^3 - 12b^4 \\
 \hline
 6a^2 b^2 - a b^3 - 12b^4
 \end{array}$$

(21.)

$$\begin{array}{r}
 x^2 + x y - y^2 \\
 x - y \\
 \hline
 x^3 + x^2 y - x y^2 \\
 - x^2 y - x y^2 + y^3 \\
 \hline
 x^3 - 2x y^2 + y^3
 \end{array}$$

(22.)

$$\begin{array}{r}
 a^2 - 4a + 16 \\
 a + 5 \\
 \hline
 a^3 - 4a^2 + 16a \\
 5a^2 - 20a + 80 \\
 \hline
 a^3 + a^2 - 4a + 80
 \end{array}$$

(23.)

$$\begin{array}{r}
 1 - a + a^2 - a^3 \\
 1 + a \\
 \hline
 1 - a + a^2 - a^3 \\
 a - a^2 + a^3 - a^4 \\
 \hline
 1 - a^4
 \end{array}$$

(24.)

$$\begin{array}{r}
 x^2 + x y + y^2 \\
 x^2 - x y + y^2 \\
 \hline
 x^4 + x^3 y + x^2 y^2 \\
 - x^3 y - x^2 y^2 - x y^3 \\
 \hline
 x^3 y^2 + x y^3 + y^4 \\
 \hline
 x^4 + x^2 y^2 + y^4
 \end{array}$$

(25.)

$$\begin{array}{r}
 a - b x \\
 c - d x \\
 \hline
 a c - b c x \\
 - a d x + b d x^2 \\
 \hline
 a c - (b c + a d) x + b d x^2
 \end{array}$$

(26.)

$$\begin{array}{r}
 3x^2 - 2x y - y^2 \\
 2x - 4y \\
 \hline
 6x^3 - 4x^2 y - 2x y^2 \\
 - 12x^2 y + 8x y^2 + 4y^3 \\
 \hline
 6x^3 - 16x^2 y + 6x y^2 + 4y^3
 \end{array}$$

(27.)

$$\begin{array}{r}
 x - y + z \\
 x + y - z \\
 \hline
 x^2 - x y + x z \\
 + x y - y^2 + y z \\
 - x z + y z - z^2 \\
 \hline
 x^2 - y^2 + 2y z - z^2
 \end{array}$$

(28.)

$$\begin{array}{r}
 27 x^3 + 9 x^2 y + 3 x y^2 + y^3 \\
 3 x - y \\
 \hline
 81 x^4 + 27 x^3 y + 9 x^2 y^2 + 3 x y^3 \\
 - 27 x^3 y - 9 x^2 y^2 - 3 x y^3 - y^4 \\
 \hline
 81 x^4 - y^4
 \end{array}$$

(29.)

$$\begin{array}{r}
 1 + x + x^4 + x^5 \\
 1 - x + x^2 - x^3 \\
 \hline
 1 + x \qquad \qquad + x^4 + x^5 \\
 - x - x^2 \qquad \qquad - x^5 - x^6 \\
 \qquad x^2 + x^3 \qquad \qquad + x^6 + x^7 \\
 \qquad \qquad - x^3 - x^4 \qquad \qquad - x^7 - x^8 \\
 \hline
 1 \qquad \qquad \qquad - x^8
 \end{array}$$

(30.)

$$\begin{array}{r}
 a^n + y^n \\
 a^n + y^n \\
 \hline
 a^{2n} + a^n y^n \\
 \qquad a^n y^n + y^{2n} \\
 \hline
 a^{2n} + 2 a^n y^n + y^{2n}
 \end{array}$$

(ART. 65, pp. 44, 45.)

(1.)

$$\begin{array}{r}
 a - b \\
 a - b \\
 \hline
 a^2 - a b \\
 - a b + b^2 \\
 \hline
 a^2 - 2 a b + b^2
 \end{array}$$

(2.)

$$\begin{array}{r}
 a + b \\
 c + d \\
 \hline
 a c + b c \\
 \qquad + a d + b d \\
 \hline
 a c + b c + a d + b d
 \end{array}$$

(3.)

$$\begin{array}{r}
 a + b \\
 a + b \\
 \hline
 a^2 + ab \\
 + ab + b^2 \\
 \hline
 a^2 + 2ab + b^2 \\
 a + b \\
 \hline
 a^3 + 2a^2b + ab^2 \\
 + a^2b + 2ab^2 + b^3 \\
 \hline
 a^3 + 3a^2b + 3ab^2 + b^3
 \end{array}$$

(4.)

$$\begin{array}{r}
 a + b \\
 a + b \\
 \hline
 a^2 + ab \\
 + ab + b^2 \\
 \hline
 a^2 + 2ab + b^2 \\
 a - b \\
 \hline
 a^3 + 2a^2b + ab^2 \\
 - a^2b - 2ab^2 - b^3 \\
 \hline
 a^3 + a^2b - ab^2 - b^3
 \end{array}$$

(5.)

$$\begin{array}{r}
 x^3 - xy + y^3 \\
 x + y \\
 \hline
 x^3 - x^2y + xy^3 \\
 + x^2y - xy^2 + y^3 \\
 \hline
 x^3 + y^3
 \end{array}$$

(6.)

$$\begin{array}{r}
 a + b + c \\
 a - b - c \\
 \hline
 a^2 + ab + ac \\
 - ab - b^2 - bc \\
 - ac - bc - c^2 \\
 \hline
 a^2 - b^2 - 2bc - c^2
 \end{array}$$

(7.)

$$\begin{array}{r}
 2x + 3 \\
 2x - 3 \\
 \hline
 4x^2 + 6x \\
 - 6x - 9 \\
 \hline
 4x^2 - 9 \\
 4x^2 + 9 \\
 \hline
 16x^4 - 36x^2 \\
 + 36x^2 - 81 \\
 \hline
 16x^4 - 81
 \end{array}$$

(8.)

$$\begin{array}{r}
 a^n + b^n \\
 a - b \\
 \hline
 a^{n+1} + ab^n \\
 - a^n b - b^{n+1} \\
 \hline
 a^{n+1} + ab^n - a^n b - b^{n+1}
 \end{array}$$

(9.)

$$\begin{array}{r}
 4 a^m + 6 b^n \\
 a^m - b^3 \\
 \hline
 4 a^{2m} + 6 a^m b^n \\
 \phantom{4 a^{2m} + 6 a^m b^n} - 4 a^m b^3 - 6 b^{n+3} \\
 \hline
 4 a^{2m} + 6 a^m b^n - 4 a^m b^3 - 6 b^{n+3}
 \end{array}$$

DIVISION.

(ART. 72, pp. 48, 49.)

$$6. \quad \frac{-16 a^3 b^3}{-8 a^3 b} = \frac{2 b}{a^2} = 2 a^{-2} b.$$

$$8. \quad \text{Ans. } 7 m.$$

$$11. \quad \text{Ans. } -3 a.$$

$$18. \quad \frac{36 a^7 b^5 c^2}{9 a^5 b^3 c^4} = \frac{4 a^2 b^2}{c^2} = 4 a^2 b^2 c^{-2}.$$

$$21. \quad \frac{27 a b^3 (x+y)^3}{3 b^3 (x+y)^3} = \frac{9 a}{b (x+y)^3} = 9 a b^{-1} (x+y)^{-3}.$$

The negative exponents found in the above answers may be obtained at once by subtracting the exponents of the divisor from the corresponding ones of the dividend; or, they may be obtained, as above, by transferring the remaining factors of the divisor to the dividend, the sign of their exponents being changed.

(ART. 73, pp. 50, 51.)

$$5. \quad \text{Ans. } 3 a^2 b + 5 a b - 2 c.$$

$$6. \quad \text{Ans. } 5 + 10 x - 12 b^2.$$

$$16. \quad 3 a + (a + b) = 4 a + b.$$

2 *

$$17. \quad 4 - (x + 3) = 4 - x - 3 = 1 - x.$$

18. The exponent of a in $5 a^2 x$ is 1, which, subtracted from -2 , gives -3 as the exponent of a in the first term of the quotient. The exponent -1 , of $2 b^{-1}$, is obtained in a similar manner, by subtracting the exponent 2 from the exponent 1. The answer may also be thus expressed :

$$3 \frac{b}{a^3} - \frac{2}{b} - 1.$$

(ART. 74, pp. 51-55.)

(4.)

$$\begin{array}{r} a + b \overline{) a^3 - b^3 (a - b} \\ \underline{a^3 + a b} \\ - a b - b^3 \\ \underline{- a b - b^3} \end{array}$$

(5.)

$$\begin{array}{r|l} a^3 + 2 a^2 b + 2 a b^2 + b^3 & a^2 + a b + b^2 \\ a^3 + + + & a + b \\ \hline a^2 b + + + & \\ a^2 b + + + & \\ \hline \end{array}$$

(6.)

$$\begin{array}{r|l} a^3 - 3 a^2 b + 3 a b^2 - b^3 & a - b \\ a^3 - + + & a^2 - 2 a b + b^2 \\ \hline - 2 a^2 b + 3 a b^2 & \\ - 2 a^2 b + 2 a b^2 & \\ \hline a b^2 - b^3 & \\ a b^2 - b^3 & \\ \hline \end{array}$$

(7.)

$$\begin{array}{r}
 a-1) a^3-1 \quad (a^2+a+1 \\
 \underline{a^3-a^2} \\
 a^2-1 \\
 \underline{a^2-a} \\
 a-1 \\
 \underline{a-1} \\
 0
 \end{array}$$

(8.)

$$\begin{array}{r|l}
 8a^3-4a^2b-6ab^2+3b^3 & 2a-b \\
 8a^3-4a^2b & \underline{4a^2-3b^2} \\
 \hline
 -6ab^2+3b^3 & \\
 -6ab^2+3b^3 & \\
 \hline
 0 &
 \end{array}$$

(9.)

$$\begin{array}{r}
 x^2-2x+3) x^4+4x^3+3x^2 \quad (x^2+2x+1 \\
 \underline{x^4-2x^3+3x^2} \\
 2x^3-3x^2+4x \\
 \underline{2x^3-4x^2+6x} \\
 x^2-2x+3 \\
 \underline{x^2-2x+3} \\
 0
 \end{array}$$

(10.)

$$\begin{array}{r}
 a^2+ax+x^2) a^3-x^3 \quad (a-x \\
 \underline{a^3+a^2x+ax^2} \\
 -a^2x-ax^2-x^3 \\
 \underline{-a^2x-ax^2-x^3} \\
 0
 \end{array}$$

(11.)

$$\begin{array}{r}
 a^2 - 2ab + 4b^2 \quad a^4 + 4a^2b^2 + 16b^4 \quad (a^2 + 2ab + 4b^2) \\
 \underline{a^4 - 2a^2b + 4a^2b^2} \\
 2a^2b + 16b^4 \\
 2a^2b - 4a^2b^2 + 8ab^3 \\
 \hline
 4a^2b^3 - 8ab^3 + 16b^4 \\
 4a^2b^3 - 8ab^3 + 16b^4 \\
 \hline
 \end{array}$$

(12.)

$$\begin{array}{r}
 a - x \quad a^3 - x^3 \quad (a^2 + ax + x^2) \\
 \underline{a^3 - a^2x} \\
 a^2x - x^3 \\
 a^2x - ax^2 \\
 \hline
 ax^2 - x^3 \\
 ax^2 - x^3 \\
 \hline
 \end{array}$$

(13.)

$$\begin{array}{r}
 a + x \quad a^3 + x^3 \quad (a^2 - ax + x^2) \\
 \underline{a^3 + a^2x} \\
 -a^2x + x^3 \\
 -a^2x - ax^2 \\
 \hline
 ax^2 + x^3 \\
 ax^2 + x^3 \\
 \hline
 \end{array}$$

(14.)

$$\begin{array}{r|l}
 x^3 - 5x^2 - 46x - 40 & x + 4 \\
 \underline{x^3 + 4x^2} & \underline{x^3 - 9x - 10} \\
 -9x^2 - 46x & \\
 -9x^2 - 36x & \\
 \hline
 -10x - 40 & \\
 -10x - 40 & \\
 \hline
 \end{array}$$

(15.)

$$\begin{array}{r|l}
 x^7 - 1 & x - 1 \\
 x^7 - x^6 & x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 \\
 \hline
 x^6 - 1 & \\
 x^6 - x^5 & \\
 \hline
 x^5 - 1 & \\
 x^5 - x^4 & \\
 \hline
 x^4 - 1 & \\
 x^4 - x^3 & \\
 \hline
 x^3 - 1 & \\
 x^3 - x^2 & \\
 \hline
 x^2 - 1 & \\
 x^2 - x & \\
 \hline
 x - 1 & \\
 x - 1 & \\
 \hline
 \end{array}$$

(16.)

$$\begin{array}{r}
 a + x) a^5 + x^5 \quad (a^4 - a^3 x + a^2 x^2 - a x^3 + x^4 \\
 \underline{a^5 + a^4 x} \\
 - a^4 x + x^5 \\
 \underline{- a^4 x - a^3 x^2} \\
 a^3 x^2 + x^5 \\
 \underline{a^3 x^2 + a^2 x^3} \\
 - a^2 x^3 + x^5 \\
 \underline{- a^2 x^3 - a x^4} \\
 a x^4 + x^5 \\
 \underline{a x^4 + x^5} \\
 \hline
 \end{array}$$

(17.)

$$\begin{array}{r|l}
 21 a^5 - 21 b^5 & 7 a - 7 b \\
 21 a^5 - 21 a^4 b & 3 a^4 + 3 a^3 b + 3 a^2 b^2 + 3 a b^3 + 3 b^4 \\
 \hline
 21 a^4 b - 21 b^5 & \\
 21 a^4 b - 21 a^3 b^2 & \\
 \hline
 & 21 a^3 b^2 - 21 b^5 \\
 & 21 a^3 b^2 - 21 a^2 b^3 \\
 \hline
 & 21 a^2 b^3 - 21 b^5 \\
 & 21 a^2 b^3 - 21 a b^4 \\
 \hline
 & 21 a b^4 - 21 b^5 \\
 & 21 a b^4 - 21 b^5 \\
 \hline
 \end{array}$$

(18.)

$$\begin{array}{r|l}
 2 a^4 + 2 a^3 b + 5 a^2 b^2 - 6 a b^3 + 4 b^4 & 2 a^2 - 2 a b + b^2 \\
 2 a^4 - 2 a^3 b + a^2 b^2 & a^2 + 2 a b + 4 b^2 \\
 \hline
 4 a^3 b + 4 a^2 b^2 - 6 a b^3 & \\
 4 a^3 b - 4 a^2 b^2 + 2 a b^3 & \\
 \hline
 & 8 a^2 b^2 - 8 a b^3 + 4 b^4 \\
 & 8 a^2 b^2 - 8 a b^3 + 4 b^4 \\
 \hline
 \end{array}$$

(19.)

$$\begin{array}{r|l}
 2 a^{m+1} - 2 a^{n+1} - a^{m+n} + a^{2n} & 2 a - a^n \\
 2 a^{m+1} & - a^m - a^n \\
 \hline
 - 2 a^{n+1} & + a^{2n} \\
 - 2 a^{n+1} & + a^{2n} \\
 \hline
 \end{array}$$

(20.)

$$\begin{array}{r}
 a+x) \ a^4 - 3x^4 \ (a^3 - a^2x + ax^2 - x^3 - \frac{2x^4}{a+x}) \\
 \underline{a^4 + a^3x} \\
 \quad -a^3x - 3x^4 \\
 \quad \underline{-a^3x - a^2x^2} \\
 \qquad \qquad a^2x^2 - 3x^4 \\
 \qquad \qquad \underline{a^2x^2 + ax^3} \\
 \qquad \qquad \qquad -ax^3 - 3x^4 \\
 \qquad \qquad \qquad \underline{-ax^3 - x^4} \\
 \qquad \qquad \qquad \qquad -2x^4
 \end{array}$$

(21.)

$$\begin{array}{r}
 a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5 \quad a^3 - 2ab + b^2 \\
 \underline{a^5 - 2a^4b + \quad a^3b^2} \quad \underline{a^3 - 3a^2b + 3ab^2 - b^3} \\
 \quad -3a^4b + 9a^3b^2 - 10a^2b^3 \\
 \quad -3a^4b + 6a^3b^2 - 3a^2b^3 \\
 \hline
 \quad \quad 3a^3b^2 - 7a^2b^3 + 5ab^4 \\
 \quad \quad 3a^3b^2 - 6a^2b^3 + 3ab^4 \\
 \hline
 \quad \quad \quad -a^2b^3 + 2ab^4 - b^5 \\
 \quad \quad \quad -a^2b^3 + 2ab^4 - b^5 \\
 \hline
 \end{array}$$

THEOREMS.

TEACHERS will observe that the principles of Involution, for the *second power* only, are here anticipated to some extent. They will also readily perceive that it is important to introduce these principles at this early stage, on account of their use in the operations connected with factoring and fractions.

The pupil already understands that a quantity is *squared*

when it is multiplied by itself (Art. 20) ; and that, so far as the literal part of a monomial is concerned, this is effected by adding the exponent of each letter to itself (Art. 62), or, which is the same thing, multiplying it by 2.

THEOREM I.

(ART. 76, pp. 55, 56.)

2. $(2x + y)^2 = (2x)^2 + 2(2x)y + y^2 = 4x^2 + 4xy + y^2.$
3. $(6a^2 + 2a^2b)^2 = (6a^2)^2 + 2(6a^2)(2a^2b) + (2a^2b)^2$
 $= 36a^4 + 24a^4b + 4a^4b^2.$
4. $(a^3b^2 + 3a^2b^3c^4)^2 = (a^3b^2)^2 + 2(a^3b^2)(3a^2b^3c^4)$
 $+ (3a^2b^3c^4)^2 = a^6b^4 + 6a^5b^5c^4 + 9a^4b^6c^8.$

THEOREM II.

(ART. 77, p. 56.)

2. $(5c - 1)^2 = (5c)^2 - 2(5c) + 1 = 25c^2 - 10c + 1.$
3. $(a^2 - b^2)^2 = (a^2)^2 - 2a^2b^2 + (b^2)^2 = a^4 - 2a^2b^2 + b^4.$
4. $(5a^2b^2 - 10a^2b^3)^2 = (5a^2b^2)^2 - 2(5a^2b^2)(10a^2b^3)$
 $+ (10a^2b^3)^2 = 25a^4b^4 - 100a^4b^5 + 100a^4b^6.$

THEOREM III.

• (ART. 78, p. 57.)

1. $(3a + 2b)(3a - 2b) = (3a)^2 - (2b)^2 = 9a^2 - 4b^2.$
3. $(5a + b)(5a - b) = (5a)^2 - b^2 = 25a^2 - b^2.$
4. $(9x + 1)(9x - 1) = (9x)^2 - 1 = 81x^2 - 1.$
5. $(3a^2c + 10ab^3)(3a^2c - 10ab^3) = (3a^2c)^2 - (10ab^3)^2$
 $= 9a^4c^2 - 100a^2b^6.$
6. $(3x^2y + 12xy^2)(3x^2y - 12xy^2) = (3x^2y)^2 - (12xy^2)^2$
 $= 9x^4y^2 - 144x^2y^4.$

MISCELLANEOUS EXAMPLES.

(pp. 57, 58.)

2. $(3a - 2)^2 = (3a)^2 - 2(3a)2 + 2^2 = 9a^2 - 12a + 4.$
3. $(9ab + 2b^2)^2 = (9ab)^2 + 2(9ab)(2b^2) + (2b^2)^2$
 $= 81a^2b^2 + 36ab^3 + 4b^4.$
5. $(2 - x^m)^2 = 2^2 - 2(2x^m) + (x^m)^2 = 4 - 4x^m + x^{2m}.$
6. $(a^3 + 1)(a^3 - 1) = (a^3)^2 - 1 = a^6 - 1.$
7. $(a^2 - b^2)(a^2 + b^2) = (a^2)^2 - (b^2)^2 = a^4 - b^4.$
8. $(1 - 3c^2)^2 = 1 - 2(3c^2) + (3c^2)^2 = 1 - 6c^2 + 9c^4.$
9. $(2 + \overline{a - b})(2 - \overline{a - b}) = 2^2 - \overline{a - b}^2 = 4 - (a - b)^2.$
10. $2(a + b)(a - b) = 2(a^2 - b^2) = 2a^2 - 2b^2.$
11. $3^3(x^2 - a^2)^2 = 3^3(x^2)^2 - (3^3)2a^2x^2 + 3^3(a^2)^2$
 $= 27x^4 - 54a^2x^2 + 27a^4.$
12. $(1 - 4a)(1 - 4a) = 1 - 2(4a) + (4a)^2$
 $= 1 - 8a + 16a^2.$
13. $(3m + 4n)(3m - 4n) = (3m)^2 - (4n)^2$
 $= 9m^2 - 16n^2.$
14. $(3a - 4x)(3a + 4x) = (3a)^2 - (4x)^2$
 $= 9a^2 - 16x^2, \text{ Ans.}$
15. $(2a + 3x)(2a + 3x) = (2a)^2 + 2(2a)(3x)$
 $+ (3x)^2 = 4a^2 + 12ax + 9x^2.$
16. $(2ac - 3bc)(2ac - 3bc) = (2ac)^2 - 2(2ac)(3bc)$
 $+ (3bc)^2 = 4a^2c^2 - 12abc^2 + 9b^2c^2.$
17. $(3 - \overline{a + b})(3 - \overline{a + b}) = 3^2 - 2(3)(a + b)$
 $+ (a + b)^2 = 9 - 6(a + b) + (a + b)^2.$
18. $(5a^2b^2 + 7ab)(5a^2b^2 - 7ab) = (5a^2b^2)^2 - (7ab)^2$
 $= 25a^4b^4 - 49a^2b^2.$

$$19. (x + a)(x - a)(x^2 - a^2) = (x^2 - a^2)(x^2 - a^2) \\ = x^4 - 2a^2x^2 + a^4.$$

$$20. (x + 2)(x - 2)(x - 3)(x + 3) = (x^2 - 4)(x^2 - 9) \\ = x^4 - 13x^2 + 36.$$

$$21. (2x + 3)(2x - 3)(4x^2 + 9) = (4x^2 - 9)(4x^2 + 9) \\ = 16x^4 - 81.$$

$$22. (x^2 - 1)(x^2 + 1)(x^4 - 1) = (x^4 - 1)(x^4 - 1) \\ = x^8 - 2x^4 + 1.$$

F A C T O R I N G.

(ART. 88, pp. 60, 61.)

3. Ans. $3 \times 7 m m m n n x$.

5. Ans. $2 \times 2 \times 2 \times 7 a a a b b b b c c x x x y$.

(ART. 89, pp. 61, 62.)

4. Ans. $x(a + 1)$.

6. Ans. $7b^2c^2x(2c^3 - 3bc + 1)$.

(ART. 90, pp. 62, 63.)

$$3. 4a^2 + 12ab + 9b^2 = (2a)^2 + 2(2a)(3b) + (3b)^2 \\ = (2a + 3b)^2 = (2a + 3b)(2a + 3b).$$

4. Ans. $(2a - 3b)(2a - 3b)$.

$$5. a^2 - 4ab^2 + 4b^4 = a^2 - 2a(2b^2) + (2b^2)^2 = (a - 2b^2)^2.$$

$$7. 1 + 2x^2 + x^4 = 1 + 2x^2 + (x^2)^2 = (1 + x^2)^2.$$

Ans. $(1 + x)(1 + x)$.

$$8. 4x^2 + 4xy + y^2 = (2x)^2 + 2(2x)y + y^2 = (2x + y)^2.$$

$$9. 25m^4 + 10m^2n + n^2 = (5m^2)^2 + 2(5m^2)n + n^2 \\ = (5m^2 + n)(5m^2 + n).$$

(ART. 91, pp. 63, 64.)

3. Ans. $(x + y)(x - y)$.

4. $4x^2 - y^2 = (2x)^2 - y^2 = (2x + y)(2x - y)$.

5. $9a^2 - 4b^2 = (3a)^2 - (2b)^2 = (3a + 2b)(3a - 2b)$.

6. $64a^2b^2 - 16c^2d^2 = (8ab)^2 - (4cd)^2$.

Ans. $(8ab + 4cd)(8ab - 4cd)$.

7. $1 - 81x^2 = 1 - (9x)^2 = (1 + 9x)(1 - 9x)$.

8. $c^6 - a^4y^2 = (c^3)^2 - (a^2y)^2 = (c^3 + a^2y)(c^3 - a^2y)$.

10. $1 - c^4 = (1 + c^2)(1 - c^2) = (1 + c^2)(1 + c)(1 - c)$.

11. $16y^8 - 1 = (4y^4 + 1)(4y^4 - 1) = (4y^4 + 1)(2y^2 + 1)(2y^2 - 1)$.

12. $a^8 - c^8 = (a^4 + c^4)(a^4 - c^4) = (a^4 + c^4)(a^2 + c^2)(a^2 - c^2) = (a^4 + c^4)(a^2 + c^2)(a + c)(a - c)$.

13. $x^8 - y^4 = (x^4 + y^2)(x^4 - y^2) = (x^4 + y^2)(x^2 + y)(x^2 - y)$.

(ART. 92, pp. 64, 65.)

2. See Art. 87.

3. Using m for a and n for b , we obtain the first answer from Art. 85, and the second from Art. 86. The same results may also be obtained by actually dividing $m^4 - n^4$ by $m - n$ and by $m + n$.

Also $m^3 + m^2n + mn^2 + n^3 = (m + n)(m^2 + n^2)$,

and $m^3 - m^2n + mn^2 - n^3 = (m - n)(m^2 + n^2)$;

hence, $m^4 - n^4 = (m - n)(m + n)(m^2 + n^2)$.

4. Substituting 1 for a and x for b , the quantity $1 - x^4$ is factored precisely as in the last example. If the second factors be again separated, we obtain the prime factors, thus, $1 - x^4 = (1 - x) (1 + x) (1 + x^2)$.

5. $8x^3 - y^3 = (2x)^3 - y^3 = (2x - y) (4x^2 + 2xy + y^2)$, by Art. 85, or by actual division.

6. $8x^3 + 1 = (2x)^3 + 1 = (2x + 1) (4x^2 - 2x + 1)$, by Art. 87, or by actual division.

7. See Art. 87.

8. The first form of the answer is obtained by Art. 91; this is changed to the second form by Art. 85, and to the third by Art. 87. This latter contains the *prime* factors.

By means of Arts. 86 and 85, or by actual division, we can also obtain

$$a^6 - b^6 = (a + b) (a^5 - a^4 b + a^3 b^2 - a^2 b^3 + a b^4 - b^5),$$

$$\text{and } a^6 - b^6 = (a - b) (a^5 + a^4 b + a^3 b^2 + a^2 b^3 + a b^4 + b^5).$$

The last factor of each expression is capable of being separated into three others, thus making up the four prime factors,

$$(a + b) (a - b) (a^2 + a b + b^2) (a^2 - a b + b^2).$$

(ART. 93, p. 66.)

$$4. \quad 6x^3 + 12x^2y + 6xy^2 = 6x(x^2 + 2xy + y^2)$$

$$= 6x(x + y)(x + y).$$

$$6. \quad ac - bd + bc - ad = a(c - d) + b(c - d)$$

$$= (a + b)(c - d).$$

$$7. \quad 6ax - 2by + 3bx - 4ay = 2a(3x - 2y)$$

$$+ b(3x - 2y) = (2a + b)(3x - 2y).$$

GREATEST COMMON DIVISOR.

(ART. 98, pp. 68, 69.)

The examples of Case I. are readily solved mentally.

4. Ans. $5 a b c d^2$.

(ART. 99, pp. 69–72.)

4. According to Note 3, we suppress the monomial factor 3 in $3x^3 - 24x - 9$, and 2 in $2x^3 - 16x - 6$, when each becomes $x^3 - 8x - 3$, which is therefore their greatest common divisor.

5. According to Note 4, we must multiply the dividend by some quantity, not a factor of the divisor, which will render its first term divisible by the first term of the divisor. As the two expressions are of the *same degree*, it is immaterial which is made the divisor (Note 1).

$$\begin{array}{r}
 6a^2 + 7ax - 3x^2 \quad | \quad 4a^2 - 4ax - 15x^2 \\
 \hline
 2 \quad | \quad (3 \\
 12a^2 + 14ax - 6x^2 \\
 12a^2 - 12ax - 45x^2 \\
 \hline
 26ax + 39x^2 \\
 2a + 3x) \quad 4a^2 - 4ax - 15x^2 \quad (2a - 5x \\
 4a^2 + 6ax \\
 \hline
 - 10ax - 15x^2 \\
 - 10ax - 15x^2 \\
 \hline
 0
 \end{array}$$

The factor $13x$ is suppressed in the first remainder (Note 3), and we thus obtain $2a + 3x$, which proves to be the greatest common divisor.

$$\begin{array}{r}
 (6.) \\
 a^2 - 2ab - 3b^2 \quad 2a^2 + ab - b^2 \quad (2) \\
 \underline{2a^2 - 4ab - 6b^2} \\
 5ab + 5b^2 \\
 a + b \quad a^2 - 2ab - 3b^2 \quad (a - 3b) \\
 \underline{a^2 + ab} \\
 a + b \quad 3ac + 3bc \quad (3c) \\
 \underline{3ac + 3bc} \qquad \qquad \underline{-3ab - 3b^2} \\
 \qquad \qquad \qquad \underline{-3ab - 3b^2}
 \end{array}$$

We find the greatest common divisor of the first two given quantities (Note 6), which is $a + b$, and then find the greatest common divisor of $a + b$ and the third given quantity.

$$\begin{array}{r}
 (7.) \\
 x^3 + 3x^2 - 4x \quad 2x^4 - 7x^3 + 5x^2 \quad (2x) \\
 \underline{2x^4 + 6x^3 - 8x^2} \\
 -13x^3 + 13x^2 \\
 x^2 - x \quad x^3 + 3x^2 - 4x \quad (x + 4) \\
 \underline{x^3 - x^2} \\
 4x^2 - 4x \\
 \underline{4x^2 - 4x}
 \end{array}$$

The factor $13x$ is suppressed in the first remainder, according to Note 3, and the signs are changed, according to Note 5. The factor $13x^2$ is found in both terms of the remainder; but a portion of this, x , is also found in the divisor, and we suppress only that part which is not common to the divisor and remainder.

We might have suppressed x from the original dividend, as we have done in the remainder; or, we might suppress x^2 from one and x from the other, if we finally restore the common factor x to the greatest common divisor (Note 2).

The first division, as given above, might be continued; for the first term of the divisor, x^3 , is contained — 13 times in the first term of the remainder, — $13 x^3$. The result would, however, be the same, after suppressing the factor found in the remainder.

(ART. 100, pp. 72, 73.)

$$2. \quad a b + b^2 = b (a + b),$$

$$a c^2 + b c^2 = c^2 (a + b).$$

Hence $a + b$, it being the only common factor, is the greatest common divisor. (Art. 96.)

$$3. \quad a^2 - 2 a = a (a - 2),$$

$$a b - 2 b = b (a - 2).$$

Hence $a - 2$ is the greatest common divisor.

$$4. \quad a^5 - a^3 b^2 = a^3 (a^2 - b^2),$$

$$a^4 - b^4 = (a^2 + b^2) (a^2 - b^2).$$

Hence $a^2 - b^2$ is the greatest common divisor.

$$5. \quad a b + a m + b n + m n = a (b + m) + n (b + m) \\ = (a + n) (b + m),$$

$$b^2 n - m^2 n = n (b^2 - m^2) = n (b - m) (b + m).$$

Hence $b + m$ is the greatest common divisor.

$$6. \quad a^2 + 2 a b + b^2 = (a + b) (a + b),$$

$$a^3 - a b^2 = a (a^2 - b^2) = a (a - b) (a + b).$$

Hence $a + b$ is the greatest common divisor.

$$7. \quad 3 x^2 - 3 y^2 = 3 (x^2 - y^2) = 3 (x - y) (x + y),$$

$$3 x^2 + 6 x y + 3 y^2 = 3 (x^2 + 2 x y + y^2) = 3 (x + y) (x + y),$$

$$3 a b x + 3 a b y = 3 a b (x + y).$$

Hence $3 (x + y)$, the product of the common factors, is the greatest common divisor (Art. 96).

LEAST COMMON MULTIPLE.

(ART. 105, pp. 74-76.)

$$\begin{aligned}
 4. \quad & (x^2 - x - 12) \div (x + 3) = x - 4, \\
 & \text{and } x^2 + 6x + 9 = x(x + 3) + 9 = x(x + 3)^2; \\
 & \text{hence } \frac{(x^2 - x - 12)(x^2 + 6x + 9)}{x + 3} = x(x - 4)(x + 3)^2.
 \end{aligned}$$

(5.)

$$\begin{aligned}
 9x^3y^5 &= 3^2x^3y^5 \\
 15xy^2 &= 3 \times 5xy^2 \\
 18x^5y^6 &= 2 \times 3^2x^5y^6 \\
 \hline
 90x^5y^6 &= 2 \times 3^2 \times 5x^5y^6
 \end{aligned}$$

(6.)

$$\begin{aligned}
 4ab^2c^3 &= 2^2ab^2c^3 \\
 6a^2c^3 &= 2 \times 3a^2c^3 \\
 9ab^5d &= 3^2ab^5d \\
 \hline
 \text{Ans. } 36a^2b^5c^3d &= 2^2 \times 3^2a^2b^5c^3d
 \end{aligned}$$

(7.)

$$\begin{aligned}
 5a^3b^2 &= 5a^3b^2 \\
 10a^2c^2(a+b) &= 2 \times 5a^2c^2(a+b) \\
 \hline
 10a^3b^2c^2(a+b) &= 2 \times 5a^3b^2c^2(a+b)
 \end{aligned}$$

(8.)

$$\begin{aligned}
 ab(x+y) &= ab(x+y) \\
 ac^2(x^3+y^3) &= ac^2(x+y)(x^2-xy+y^2) \\
 \hline
 ab^2c^2(x^3+y^3) &= ab^2c^2(x+y)(x^2-xy+y^2)
 \end{aligned}$$

(9.)

$$\begin{aligned}
 3a+1 &= 3a+1 \\
 3(9a^2-1) &= 3(3a+1)(3a-1) \\
 \hline
 3(9a^2-1) &= 3(3a+1)(3a-1)
 \end{aligned}$$

(10.)

$$1 + a = 1 + a$$

$$1 - a = 1 - a$$

$$1 - a^2 = (1 + a)(1 - a)$$

$$1 - a^2 = (1 + a)(1 - a)$$

(11.)

$$a(x + y)(x - y) = a(x^2 - y^2). \quad \text{See Note 1.}$$

(12.)

$$3a^2x + 6abx + 3b^2x = 3x(a + b)^2$$

$$12a^2 - 12ab + 3b^2 = 3(2a - b)^2$$

$$3x(a + b)^2(2a - b)^2$$

FRACTIONS.

REDUCTION.

(ART. 124, pp. 80, 81.)

5. Ans. $\frac{3x}{8}$.

$$8. \frac{a^3 - ab^2}{a^3 + 2ab + b^3} = \frac{a(a-b)(a+b)}{(a+b)(a+b)} = \frac{a(a-b)}{a+b} = \frac{a^2 - ab}{a+b}.$$

$$9. \frac{x^2 - 1}{2xy + 2y} = \frac{(x-1)(x+1)}{2y(x+1)} = \frac{x-1}{2y}.$$

$$10. \frac{ax + x^3}{a^2c^2 + c^2x} = \frac{x(a+x)}{c^2(a+x)} = \frac{x}{c^2}.$$

$$11. \frac{x^3 - a^2x}{x^2 + 2ax + a^2} = \frac{x(x-a)(x+a)}{(x+a)(x+a)} = \frac{x(x-a)}{x+a}.$$

(ART. 125, pp. 82, 83.)

$$3. \quad \frac{x^2 - y^2}{x + y} = \frac{(x + y)(x - y)}{x + y} = x - y.$$

(4.)

$$\frac{a + x) \quad ax + 2x^2 \quad (x + \frac{x^2}{a + x})}{\quad \quad \quad \frac{ax + x^2}{x^2}}$$

$$5. \quad \frac{x^2 - y^2}{x - y} = \frac{(x - y)(x^2 + xy + y^2)}{x - y} = x^2 + xy + y^2.$$

$$6. \quad \frac{12x^2 - 18}{3x} = 4x - \frac{18}{3x} = 4x - \frac{6}{x}, \text{ Ans.}$$

$$7. \quad \begin{array}{r} 2x^2 - x + 1 \overline{) 4x^2 - 2x} \quad (2 - \frac{2}{2x^2 - x + 1}) \\ \underline{4x^2 - 2x + 2} \\ -2 \end{array}$$

$$8. \quad \frac{a^2 - b^2 + x^2}{a + x} = \frac{a^2 + x^2}{a + x} - \frac{b^2}{a + x} = \frac{(a + x)(a^2 - ax + x^2)}{a + x} - \frac{b^2}{a + x} = a^2 - ax + x^2 - \frac{b^2}{a + x}.$$

(ART. 126, pp. 83, 84.)

$$3. \quad \frac{5ab^2}{3b^2c^2} = \frac{5a}{3bc^2} = 5 \times 3^{-1} a b^{-1} c^{-2}.$$

$$4. \quad \frac{3xy^2}{3x^2y} = \frac{y^2}{x} = x^{-1}y^2. \text{ Ans.}$$

$$6. \frac{x^2 y^2}{a^{-3} y^{-2}} = a^3 x^2 y^2 y^2 = a^3 x^2 y^4.$$

$$8. \frac{a-b}{(a+b)^{-1}} = (a-b)(a+b) = a^2 - b^2.$$

$$9. \frac{4 a^{-3} b^3 x^{-2}}{2 a^{-2} b^{-2}} = \frac{4 a^3 b^3 b^2}{2 a^2 x^2} = \frac{2 b^5}{a x^2}.$$

$$10. \frac{xy(a-b)^{-2}}{a+b} = \frac{xy}{(a+b)(a-b)^2} = \frac{xy}{a^3 - a^2b - ab^2 + b^3}.$$

In the above examples, factors are transferred from either term of a fraction to the other by changing the signs of their exponents.

Since the fractional form may be regarded as an expression of division, the required result may also be obtained, in each of the first eight examples of Art. 126, by simply subtracting the exponents found in the denominator from those connected with the same letters or quantities in the numerator. The exponent of any letter or quantity not found in the numerator may be regarded as 0. (Art. 70.)

(ART. 127, pp. 84-86.)

$$4. \text{ Ans. } \frac{ab + a^2 x^2}{a}.$$

$$\begin{aligned} 5. \quad a - \frac{ab - a^2}{2b} &= \frac{2ab - (ab - a^2)}{2b} = \frac{2ab - ab + a^2}{2b} \\ &= \frac{ab + a^2}{2b}. \end{aligned}$$

$$6. \quad a + 1 - \frac{x-1}{b} = \frac{ab + b - (x-1)}{b} = \frac{ab + b - x + 1}{b}.$$

$$7. \quad 2a - 2b + \frac{a-x}{3} = \frac{6a - 6b + a - x}{3} = \frac{7a - 6b - x}{3}.$$

$$8. \quad 1 + 3a - \frac{4x-5}{4x} = \frac{4x + 12ax - (4x-5)}{4x} = \frac{12ax + 5}{4x}.$$

$$9. \quad a + b - \frac{a^2 - b^2 - 3}{a-b} = \frac{a^2 - b^2 - (a^2 - b^2 - 3)}{a-b} = \frac{3}{a-b}.$$

$$10. \quad 2 + \frac{x^2 + y^2}{xy} = \frac{2xy + x^2 + y^2}{xy} = \frac{(x+y)^2}{xy}.$$

$$11. \quad a + b - \frac{a^2 - 2ab + b^2}{a+b} = \frac{a^2 + 2ab + b^2 - (a^2 - 2ab + b^2)}{a+b} \\ = \frac{4ab}{a+b}.$$

(ART. 128, pp. 86-88.)

4. The least common multiple of the denominators $5y$, $10y^2$, and $2x$, is $10xy^2$. (ART. 105.)

$$(10xy^2 \div 5y) \times 4x = 8x^2y; \quad \frac{4x}{5y} = \frac{8x^2y}{10xy^2}.$$

$$(10xy^2 \div 10y^2) \times 7m = 7mx; \quad \frac{7m}{10y^2} = \frac{7mx}{10xy^2}.$$

$$(10xy^2 \div 2x) \times n = 5ny^2; \quad \frac{n}{2x} = \frac{5ny^2}{10xy^2}.$$

5. The least common multiple of $2a$, $5a^2$, and n is $10a^2n$.

$$(10a^2n \div 2a) \times 3x = 15anx; \quad \frac{3x}{2a} = \frac{15anx}{10a^2n}.$$

$$(10a^2n \div 5a^2) \times 2x = 4nx; \quad \frac{2x}{5a^2} = \frac{4nx}{10a^2n}.$$

$$(10a^2n \div n) \times m = 10a^2m; \quad \frac{m}{n} = \frac{10a^2m}{10a^2n}.$$

$$\text{Ans.} \quad \frac{15anx}{10a^2n}, \quad \frac{4nx}{10a^2n}, \quad \frac{10a^2m}{10a^2n}.$$

Examples 6 and 7 are readily solved by the first part of the rule.

8. The least common multiple of 1, b , and $c + d$ is $b(c + d)$, or $bc + bd$.

$$[(bc + bd) \div 1] \times a = abc + abd; \frac{a}{1} = \frac{abc + abd}{bc + bd}.$$

$$[(bc + bd) \div b] \times a = ac + ad; \frac{a}{b} = \frac{ac + ad}{bc + bd}.$$

$$[(bc + bd) \div (c + d)] \times (x - 2) = bx - 2b; \frac{x - 2}{c + d} = \frac{bx - 2b}{bc + bd}.$$

NOTE. The same results may also be obtained by means of the first part of the rule.

(ART. 129, p. 89.)

2. Since $(x + y)(x - y) = x^2 - y^2$, it is evident that if we multiply both numerator and denominator of the first fraction by $x - y$, and of the second by $x + y$, the denominator of each of the three fractions will be $x^2 - y^2$.

3. Since $(a - x)(a^2 + ax + x^2) = a^3 - x^3$ (Art. 85), it is evident that the multiplier $a^2 + ax + x^2$ should be used for the second fraction, and $a - x$ for the third.

ADDITION.

(ART. 131, pp. 90, 91.)

$$3. \frac{3a}{4} + \frac{5a}{6} + \frac{a}{3} = \frac{9a}{12} + \frac{10a}{12} + \frac{4a}{12} = \frac{23a}{12}.$$

$$4. \frac{x}{2} + \frac{x}{3} + \frac{x}{4} = \frac{6x}{12} + \frac{4x}{12} + \frac{3x}{12} = \frac{13x}{12}.$$

$$5. \quad \frac{3x}{2a} + \frac{x}{5} = \frac{15x}{10a} + \frac{2ax}{10a} = \frac{15x + 2ax}{10a}.$$

$$7. \quad \frac{4x}{7} + \frac{x-2}{5} = \frac{20x}{35} + \frac{7x-14}{35} = \frac{27x-14}{35}.$$

$$8. \quad \frac{3x-a}{14} + \frac{4x}{7} = \frac{3x-a}{14} + \frac{8x}{14} = \frac{11x-a}{14}, \text{ Ans.}$$

$$9. \quad \frac{x}{x+y} + \frac{y}{x-y} = \frac{x^2-xy}{x^2-y^2} + \frac{xy+y^2}{x^2-y^2} = \frac{x^2+y^2}{x^2-y^2}.$$

$$10. \quad \frac{1}{1+a} + \frac{1}{1-a} = \frac{1-a}{1-a^2} + \frac{1+a}{1-a^2} = \frac{2}{1-a^2}.$$

$$11. \quad \frac{x+y}{x-y} + \frac{x-y}{x+y} = \frac{x^2+2xy+y^2}{x^2-y^2} + \frac{x^2-2xy+y^2}{x^2-y^2} \\ = \frac{2x^2+2y^2}{x^2-y^2} = \frac{2(x^2+y^2)}{x^2-y^2}.$$

$$12. \quad \frac{a-b}{ab} + \frac{b-c}{bc} + \frac{c-a}{ac} = \frac{ac-bc}{abc} + \frac{ab-ac}{abc} + \frac{bc-ab}{abc} \\ = \frac{0}{abc} = 0.$$

$$13. \quad \frac{x-a}{x^2-ax+a^2} + \frac{1}{x+a} = \frac{x^2-a^2}{x^2+a^2} + \frac{x^2-ax+a^2}{x^2+a^2} \\ = \frac{2x^2-ax}{x^2+a^2}.$$

$$14. \quad \frac{n}{n-1} + \frac{1-2n}{n^2-n} = \frac{n^2}{n^2-n} + \frac{1-2n}{n^2-n} = \frac{n^2-2n+1}{n^2-n} \\ = \frac{(n-1)(n-1)}{n(n-1)} = \frac{n-1}{n}.$$

$$16. \quad 3a + \frac{2x}{5} + a - \frac{8x}{9} = 3a + a + \frac{18x}{45} - \frac{40x}{45} = 4a - \frac{22x}{45}.$$

$$17. \quad c + \frac{c+d}{2} + \frac{c-d}{2} = c + \frac{2c}{2} = c + c = 2c.$$

$$\begin{aligned}
 18. \quad x - \frac{4a^2}{b} + y + \frac{2ax}{c} &= x + y - \frac{4a^2c}{bc} + \frac{2abx}{bc} \\
 &= x + y + \frac{2abx - 4a^2c}{bc}.
 \end{aligned}$$

SUBTRACTION.

(ART. 133, pp. 92-94.)

$$3. \quad \frac{12x}{7} - \frac{3x}{5} = \frac{60x}{35} - \frac{21x}{35} = \frac{39x}{35}.$$

$$4. \quad \frac{a}{3} - \frac{a}{4} = \frac{4a}{12} - \frac{3a}{12} = \frac{a}{12}.$$

$$5. \quad \frac{3ab}{4} - \frac{4ab}{2} = \frac{3ab}{4} - \frac{8ab}{4} = -\frac{5ab}{4}.$$

$$6. \quad \frac{x+y}{2} - \frac{x-y}{2} = \frac{2y}{2} = y.$$

$$7. \quad \frac{1}{x-1} - \frac{1}{x+1} = \frac{x+1}{x^2-1} - \frac{x-1}{x^2-1} = \frac{2}{x^2-1}, \quad \text{Ans.}$$

$$8. \quad \frac{1-2n}{n^2-n} - \frac{n}{n-1} = \frac{1-2n}{n^2-n} - \frac{n^2}{n^2-n} = \frac{1-2n-n^2}{n^2-n}.$$

$$\begin{aligned}
 9. \quad 3x - \frac{x}{2b} - \left(x - \frac{x-a}{c}\right) &= 3x - x - \frac{cx}{2bc} + \frac{2bx-2ab}{2bc} \\
 &= 2x + \frac{2bx-2ab-cx}{2bc} = \frac{4bcx+2bx-cx-2ab}{2bc}.
 \end{aligned}$$

$$\begin{aligned}
 10. \quad \frac{1+x^2}{1-x^2} - \frac{4x^2}{1-x^4} &= \frac{(1+x^2)^2}{1-x^4} - \frac{4x^2}{1-x^4} = \frac{1+2x^2+x^4-4x^2}{1-x^4} \\
 &= \frac{1-2x^2+x^4}{1-x^4} = \frac{(1-x^2)(1-x^2)}{(1-x^2)(1+x^2)} = \frac{1-x^2}{1+x^2}.
 \end{aligned}$$

$$11. \quad 4x + \frac{b}{a} - \left(3x - \frac{c}{d}\right) = 4x - 3x + \frac{bd}{ad} + \frac{ac}{ad} \\ = x + \frac{bd + ac}{ad}.$$

$$12. \quad 7b - \frac{4a+1}{2} - \left(b + \frac{3}{5}\right) = 7b - b - \frac{20a+5}{10} - \frac{6}{10} \\ = 6b - \frac{20a+11}{10}.$$

$$13. \quad \frac{3x+2}{y} - \frac{7xy-10y}{y^2} = \frac{3x+2}{y} - \frac{7x-10}{y} \\ = \frac{3x+2-7x+10}{y} = \frac{12-4x}{y} = \frac{4(3-x)}{y}.$$

$$14. \quad 1 - \frac{x-a}{x+a} = \frac{x+a}{x+a} - \frac{x-a}{x+a} = \frac{x+a-x+a}{x+a} = \frac{2a}{x+a}.$$

$$15. \quad 2x + \frac{3x}{a} - x - \frac{2x-2a}{3c} = 2x - x + \frac{9cx}{3ac} - \frac{2ax-2a^2}{3ac} \\ = x + \frac{9cx - 2ax + 2a^2}{3ac}.$$

$$16. \quad \frac{xy}{x-y} - \frac{xy}{x+y} = \frac{x^2y + xy^2}{x^2 - y^2} - \frac{x^2y - xy^2}{x^2 - y^2} \\ = \frac{x^2y + xy^2 - x^2y + xy^2}{x^2 - y^2} = \frac{2xy^2}{x^2 - y^2}.$$

MULTIPLICATION.

(ART. 135, pp. 94, 95.)

$$8. \quad \frac{b-c}{ab+ac+bc+c^2} \times (a+c) = \frac{(b-c)(a+c)}{(b+c)(a+c)} = \frac{b-c}{b+c}.$$

$$9. \quad \frac{a+b+c}{9(x-y)(x+y)} \times 3(x+y) = \frac{3(a+b+c)(x+y)}{9(x-y)(x+y)} \\ = \frac{a+b+c}{3(x-y)}.$$

(ART. 137, pp. 96-98.)

6. Ans. $\frac{b c m^2}{a d}$.

11. $\left(a + \frac{x}{y}\right) \times \frac{y}{a} = \frac{a y + x}{y} \times \frac{y}{a} = \frac{a y + x}{a}$.

A factor occurring in any numerator will cancel the same factor found in any denominator, when the fractions are to be multiplied together.

13. $\frac{4x+2}{3} \times \frac{5x}{2x+1} = \frac{2(2x+1)}{3} \times \frac{5x}{2x+1} = \frac{10x}{3}$.

14. $\frac{a+b}{b} \times \frac{b^2 x}{a^2 - b^2} = \frac{a+b}{b} \times \frac{b^2 x}{(a+b)(a-b)} = \frac{b x}{a-b}$.

16. $\frac{a^2 + b^2}{a^2 - b^2} \times \frac{a-b}{a+b} = \frac{a^2 + b^2}{(a+b)(a-b)} \times \frac{a-b}{a+b} = \frac{a^2 + b^2}{(a+b)^2}$
 $= \frac{a^2 + b^2}{a^2 + 2ab + b^2}$.

17. $\frac{a-b}{a} \times \frac{a+b}{b} \times \frac{a^2}{a^2 - b^2} = \frac{a-b}{a} \times \frac{a+b}{b}$
 $\times \frac{a^2}{(a+b)(a-b)} = \frac{a}{b}$.

18. $\left(a - \frac{b^2}{a}\right) \left(\frac{a}{b} + \frac{b}{a}\right) = \frac{a^2 - b^2}{a} \times \frac{a^2 + b^2}{ab} = \frac{a^4 - b^4}{a^2 b}$.

19. $\left(\frac{x}{x+y}\right) \left(\frac{1}{x-y}\right) \left(\frac{x^2 - y^2}{x}\right) = \frac{x}{x+y} \times \frac{1}{x-y}$
 $\times \frac{(x+y)(x-y)}{x} = 1$.

20. $\left(a + \frac{b}{y}\right) \times \left(x + \frac{m}{n}\right) = \frac{a y + b}{y} \times \frac{n x + m}{n}$
 $= \frac{a n x y + a m y + b n x + b m}{n y}$.

DIVISION.

(ART. 139, pp. 99, 100.)

$$6. \quad \frac{x + xy}{1 + y} \div x = \frac{x(1 + y)}{1 + y} \div x = \frac{1 + y}{1 + y} = 1.$$

$$7. \quad \frac{abc + b^2c}{x} \div (a + b) = \frac{bc(a + b)}{x} \div (a + b) = \frac{bc}{x}.$$

$$8. \quad \frac{a^2d + abd}{c} \div ad = \frac{ad(a + b)}{c} \div ad = \frac{a + b}{c}, \text{ Ans.}$$

$$9. \quad \frac{b - c}{b + c} \div (a + c) = \frac{b - c}{(b + c)(a + c)} = \frac{b - c}{ab + ac + bc + c^2}.$$

(ART. 140, pp. 100-103.)

$$3. \quad \frac{3ay}{4m} \div \frac{2m}{y} = \frac{3ay}{4m} \times \frac{y}{2m} = \frac{3ay^2}{8m^2}.$$

$$4. \quad \frac{a}{2} \div \frac{3m}{4a} = \frac{a}{2} \times \frac{4a}{3m} = \frac{2a^2}{3m}.$$

$$5. \quad \frac{4a^2}{9m} \div \frac{2a}{3} = \frac{4a^2}{9m} \times \frac{3}{2a} = \frac{2a}{3m}.$$

$$6. \quad \frac{8a}{9m} \div \frac{a}{n} = \frac{8a}{9m} \times \frac{n}{a} = \frac{8n}{9m}, \text{ Ans.}$$

$$7. \quad \frac{ab}{1} \div \frac{3d}{2ab} = \frac{ab}{1} \times \frac{2ab}{3d} = \frac{2a^2b^2}{3d}.$$

$$8. \quad \frac{ay}{1} \div \frac{a - b}{x} = \frac{ay}{1} \times \frac{x}{a - b} = \frac{axy}{a - b}.$$

$$9. \quad \frac{a}{1 - a} \div \frac{a}{4} = \frac{a}{1 - a} \times \frac{4}{a} = \frac{4}{1 - a}.$$

$$10. \frac{a+b}{2} \div \frac{a-b}{2} = \frac{a+b}{2} \times \frac{2}{a-b} = \frac{a+b}{a-b}.$$

$$11. \left(x + \frac{a}{y}\right) \div \frac{y}{a} = \frac{xy+a}{y} \times \frac{a}{y} = \frac{axy+a^2}{y^2}.$$

$$12. \frac{3a+b}{6} \div \frac{ab+m}{7a} = \frac{3a+b}{6} \times \frac{7a}{ab+m} = \frac{21a^2+7ab}{6ab+6m}.$$

$$13. \frac{a^2m+bm y}{an} \div \left(a + \frac{by}{a}\right) = \frac{a^2m+bm y}{an} \div \frac{a^2+by}{a} \\ = \frac{m(a^2+by)}{an} \times \frac{a}{a^2+by} = \frac{m}{n}.$$

$$14. \frac{a+1}{a} \div \frac{a^2-1}{a} = \frac{a+1}{a} \times \frac{a}{a^2-1} = \frac{a+1}{a} \\ \times \frac{a}{(a+1)(a-1)} = \frac{1}{a-1}.$$

$$15. \frac{6x^2-2x}{4-x^2} \div \frac{x^2}{2+x} = \frac{x(6x-2)}{(2-x)(2+x)} \times \frac{2+x}{x^2} \\ = \frac{6x-2}{x(2-x)} = \frac{6x-2}{2x-x^2}.$$

$$16. \frac{2b^2}{a^2+b^2} \div \frac{b}{a+b} = \frac{2b^2}{(a+b)(a^2-ab+b^2)} \times \frac{a+b}{b} \\ = \frac{2b}{a^2-ab+b^2}.$$

$$17. \frac{x^2-y^2}{x+2y} \div \frac{x-y}{3x+6y} = \frac{(x+y)(x-y)}{x+2y} \times \frac{3(x+2y)}{x-y} \\ = 3(x+y).$$

$$18. (1+x) \div \frac{1+x}{x} = \frac{1+x}{1} \times \frac{x}{1+x} = x.$$

$$\begin{aligned}
 19. \quad 12 \div \left(\frac{(a+x)^2}{x} - a \right) &= \frac{12}{1} \div \frac{a^2 + ax + x^2}{x} \\
 &= \frac{12}{1} \times \frac{x}{a^2 + ax + x^2} = \frac{12x}{a^2 + ax + x^2}.
 \end{aligned}$$

(ART. 142, pp. 103-105.)

$$2. \quad \frac{x}{y} \div \frac{m}{n} = \frac{x}{y} \times \frac{n}{m} = \frac{nx}{my}.$$

$$3. \quad \frac{a+b}{x} \div \frac{y-a}{b} = \frac{a+b}{x} \times \frac{b}{y-a} = \frac{ab + b^2}{xy - ax}.$$

4. Multiply both numerator and denominator of the complex fraction by b , according to the second part of the rule, and the answer is obtained at once.

$$5. \quad \frac{5-c}{x} \div \frac{7-y}{a} = \frac{5-c}{x} \times \frac{a}{7-y} = \frac{5a - ac}{7x - xy}. \quad \text{Or,}$$

multiply both terms of the complex fraction by ax .

6. Such a fraction as $\frac{y - m n^{-1}}{x + a d^{-1}}$ is complex in meaning, though not in form, being equivalent to $\frac{y - \frac{m}{n}}{x + \frac{a}{d}}$. It is evident from the rule that the denominators n and d may be removed by multiplying both terms of the complex fraction by dn ; and, according to the note, the negative exponents of the original form are removed by the use of the same multiplier, thus:

$$\frac{(y - m n^{-1}) \times dn}{(x + a d^{-1}) \times dn} = \frac{dny - dm}{dnx + an}.$$

$$7. \frac{b}{x+a^{-1}} = \frac{b \times a}{(x+a^{-1})a} = \frac{ab}{ax+a^0} = \frac{ab}{ax+1}. \quad (\text{Art. 70.})$$

$$8. \frac{\frac{1}{b-x}}{\frac{1}{b+x}} = \frac{\frac{1}{b-x} \times (b-x)}{(\frac{1}{b+x}) \times (b-x)} = \frac{1}{b^2-x^2}.$$

$$9. \frac{\frac{2x^2-b}{5}}{\frac{a-b}{4}} = \frac{\frac{2x^2-b}{5} \times 20}{\frac{a-b}{4} \times 20} = \frac{4(2x^2-b)}{5(a-b)} = \frac{8x^2-4b}{5a-5b}.$$

10. The operation for reducing the given complex fraction to a simple one, according to the rule, is as follows :

$$\frac{x^4 - \frac{1}{x^4}}{x + \frac{1}{x}} = \frac{\left(x^4 - \frac{1}{x^4}\right) \times x^4}{\left(x + \frac{1}{x}\right) \times x^4} = \frac{x^8 - 1}{x^5 + x^3}.$$

The last expression may be reduced thus (Art. 86) :

$$\begin{aligned} \frac{x^8 - 1}{x^5 + x^3} &= \frac{(x^3 + 1)(x^5 - x^4 + x^2 - 1)}{x^3(x^3 + 1)} = \frac{x^5 - x^4 + x^2 - 1}{x^3} \\ &= x^2 - x + \frac{1}{x} - \frac{1}{x^3}. \end{aligned}$$

$$\text{Again, } \frac{x^4 - \frac{1}{x^4}}{x + \frac{1}{x}} = \frac{x^4 - x^{-4}}{x + x^{-1}} = x^3 - x + x^{-1} - x^{-3}. \quad \text{In}$$

performing the division, negative exponents must be subtracted or added whenever positive ones would be ; but these operations must be performed in accordance with the principles already laid down in the case of other negative quantities, thus :

$$\begin{array}{r}
 x^4 - x^{-4} \quad | \quad x + x^{-1} \\
 x^4 + x^2 \quad | \quad x^3 - x + x^{-1} - x^{-3} \\
 \hline
 -x^2 - x^{-4} \\
 -x^2 - x^0 \\
 \hline
 x^0 - x^{-4} \\
 x^0 + x^{-2} \\
 \hline
 -x^{-2} - x^{-4} \\
 -x^{-2} - x^{-4} \\
 \hline
 \end{array}$$

$$\begin{array}{r}
 x^4 - \frac{1}{x^4} \quad | \quad x + \frac{1}{x} \\
 x^4 + x^2 \quad | \quad x^3 - x + \frac{1}{x} - \frac{1}{x^3} \\
 \hline
 -x^2 - \frac{1}{x^4} \\
 -x^2 - 1 \\
 \hline
 1 - \frac{1}{x^4} \\
 1 + \frac{1}{x^2} \\
 \hline
 -\frac{1}{x^2} - \frac{1}{x^4} \\
 -\frac{1}{x^2} - \frac{1}{x^4} \\
 \hline
 \end{array}$$

Simpler

$$\frac{x^4 - \frac{1}{x^4}}{x + \frac{1}{x}} \times \frac{x^5}{x^5} = \frac{x^9 - \frac{1}{x^9}}{x^6 + \frac{1}{x^6}} = x^2 - 1$$

SIMPLE EQUATIONS.

TRANSFORMATION OF EQUATIONS.

(ART. 152, pp. 107-109.)

5. Ans. $5x - 2x = 24 - 3.$

NOTE. The student must always bear in mind the fact that transposition is simply adding to, or subtracting from, both members of an equation a quantity equal in numerical value to the one which is to be transposed, according to the first and second axioms. It may be well for the teacher to require this explanation of each transposition, until the idea is firmly fixed in the student's mind. Thus, in the fifth example, we subtract 3 and $2x$ from both members of the equation, according to Ax. 2.

(ART. 153, pp. 109-111.)

2. Multiply by a .

5. Multiply by 10.

3. Multiply by 4.

6. Multiply by x .

4. Multiply by 6.

$$\begin{array}{ll}
 7. \text{ Given} & m + x^{-1} = n - p x^{-1} \\
 \text{Multiplying by } x, & m x + x^0 = n x - p x^0 \\
 \text{Or,} & m x + 1 = n x - p
 \end{array}$$

NOTE. It will be seen that this example is precisely analogous to the sixth; but the equation takes a different form, merely because negative exponents are used instead of denominators.

The student should become accustomed to the use of negative exponents as early as possible. He will find that they may often be conveniently substituted for fractional forms; in fact, they constitute the natural algebraic expression of unperformed division, just as a negative coefficient expresses an unperformed subtraction. The ability to comprehend and employ them will also be found essential in the higher departments of mathematical analysis.

$$8. \text{ Multiply by 2. Ans. } 2a - 1 + 2b = 2c + d - 2x.$$

$$\begin{array}{ll}
 9. \text{ Given} & x - \frac{4x + 8}{6} = 8 \\
 \text{Multiplying by 6,} & 6x - (4x + 8) = 48 \\
 \text{Hence (Note 2),} & 6x - 4x - 8 = 48
 \end{array}$$

12. When the denominators are prime to each other, it is convenient to multiply each numerator by all the denominators except its own. In this instance, therefore, we may thus multiply both members of the equation by 55.

NOTE. Here, as in transposition, the student must be reminded that he can perform no operation upon one member of an equation without also performing precisely the same operation upon the other member. The process of clearing of fractions is based entirely upon the third axiom. Multiplying each numerator by all the denominators except its own is simply a convenient practical expedient for multiplying both members by the common multiple of all the denominators.

$$13. \text{ Multiply by 6, and apply Note 2.}$$

14. Multiplying both members of this equation by bc removes both denominators and the negative exponent.

15. Multiply both members of the equation by 8, and develop the expression $3(44 - x + 12)$ for the second member.

16. Multiplying both members of the equation by 9 removes all the denominators, and gives

$$6x - 60 = 4(30 - x).$$

Performing the multiplication indicated in the second member, we obtain

$$6x - 60 = 120 - 4x.$$

NOTE. In multiplying a fractional term by the least common multiple of the denominators, we may first divide this multiple by the denominator of the term, and then multiply the numerator by the quotient, as in reducing fractions to a common denominator. (Art. 128.)

17. Given $\frac{x-2}{x+2} + \frac{x+2}{x-2} = 14.$

Multiplying both members by $(x+2)(x-2)$, that is each numerator by the denominator of the other fraction, and 14 by both denominators,

$$(x-2)^2 + (x+2)^2 = 14(x+2)(x-2)$$

Or, $x^2 - 4x + 4 + x^2 + 4x + 4 = 14(x^2 - 4)$

SOLUTION OF SIMPLE EQUATIONS CONTAINING ONE UNKNOWN QUANTITY.

(ART. 159, pp. 112-118.)

5. Given $5x + 43 - 5 = 100 - 27$

Transposing the known terms to the second member,

$$5x = 100 - 27 - 43 + 5$$

Uniting the known terms, $5x = 35$

Dividing both members by 5, the coefficient of x ,

$$x = 7$$

6. Given $7x + 7 + 1 = 96 - 11$
 Transposing known terms, $7x = 96 - 11 - 7 - 1$
 Uniting terms, $7x = 77$
 Dividing by 7, $x = 11$
7. Given $15x + 8 - 9 = 212 + 87$
 Transposing terms, $15x = 212 + 87 - 8 + 9$
 Uniting terms, $15x = 300$
 Dividing by 15, $x = 20$
8. Given $9x + 9 = x - 71$
 Transposing terms, $9x - x = -71 - 9$
 Uniting terms, $8x = -80$
 Dividing by 8, $x = -10$
9. Given $4x - 15 = 2x + 13$
 Transposing terms, $4x - 2x = 13 + 15$
 Uniting terms, $2x = 28$
 Dividing by 2, $x = 14$
10. Given $4(x - 12) = 2(12 - x)$
 Expanding, $4x - 48 = 24 - 2x$
 Transposing terms, $4x + 2x = 24 + 48$
 Uniting terms, $6x = 72$
 Dividing by 6, $x = 12$
11. Given $9(x + 1) = 12(x - 2)$
 Expanding, $9x + 9 = 12x - 24$
 Transposing terms, $9x - 12x = -24 - 9$
 Uniting terms, $-3x = -33$
 Dividing by -3 , $x = 11$
12. Given $3(x - 3) + 2x = 3(40 - x - 19)$
 Expanding, $3x - 9 + 2x = 120 - 3x - 57$
 Transposing, $3x + 2x + 3x = 120 - 57 + 9$
 Uniting, $8x = 72$
 Dividing by 8, $x = 9$

13. Given $3(2x + 3x) - 15 = 72 - 4(x - 2)$
 Expanding, $6x + 9x - 15 = 72 - 4x + 8$
 Transposing, $6x + 9x + 4x = 72 + 8 + 15$
 Uniting, $19x = 95$
 Dividing by 19, $x = 5$

14. Given $2(x - 6) + 3(2x + 5) = 3(3x - 2) - 1$
 Expanding, $2x - 12 + 6x + 15 = 9x - 6 - 1$
 Transposing, $2x + 6x - 9x = -6 - 1 + 12 - 15$
 Uniting, $-x = -10$
 Changing signs, $x = 10$ (Art. 152, Note.)

15. Given $x - \frac{x}{2} - \frac{x}{6} = 30$
 Clearing of fractions by multiplying by 6,
 $6x - 3x - x = 180$
 Uniting terms, $2x = 180$
 Dividing by 2, $x = 90$

16. FIRST SOLUTION.

Given $1 - 8x^{-1} = \frac{1}{5} + 8x^{-1}$
 Multiplying by $5x$, to remove denominators and negative exponents, $5x - 40 = x + 40$
 Transposing terms, $5x - x = 40 + 40$
 Uniting terms, $4x = 80$
 Dividing by 4, $x = 20$

SECOND SOLUTION.

Given $1 - 8x^{-1} = \frac{1}{5} + 8x^{-1}$
 Multiplying by 5, $5 - 40x^{-1} = 1 + 40x^{-1}$
 Transposing, $-40x^{-1} - 40x^{-1} = 1 - 5$
 Uniting, $-80x^{-1} = -4$
 Dividing by -80 , $x^{-1} = \frac{4}{80} = \frac{1}{20}$
 Taking the reciprocals, $x = 20$

17. Given $\frac{2}{3}x + 12 = \frac{4}{5}x + 6$
 Multiplying by 15, $10x + 180 = 12x + 90$
 Transposing, $10x - 12x = 90 - 180$
 Uniting, $-2x = -90$
 Dividing by -2 , $x = 45$

18. Given $\frac{x}{3} + \frac{3x}{5} + \frac{4x}{7} = 158$
 Multiplying by 105, $35x + 63x + 60x = 16590$
 Uniting, $158x = 16590$
 Dividing by 158, $x = 105$

NOTE. In clearing of fractions, the multiplication of the second member need only be expressed, thus: 158×105 . The factor 158 will then be removed by the next division, and leave the other factor, 105, as the value of x .

19. Given $\frac{x}{2} + \frac{x}{4} + \frac{x}{8} = 28$
 Multiplying by 8, $4x + 2x + x = 224$
 Uniting, $7x = 224$
 Dividing by 7, $x = 32$

21. Given $\frac{ax}{n} = d$
 Multiplying by n , $ax = nd$
 Dividing by a , $x = \frac{nd}{a}$

22. Given $\frac{ax}{2} + \frac{bx}{3} = c$
 Multiplying by 6, $3ax + 2bx = 6c$
 Factoring, $(3a + 2b)x = 6c$
 Dividing by $3a + 2b$, $x = \frac{6c}{3a + 2b}$

23. Given $\frac{1}{a} + \frac{b}{x} = c$
 Multiplying by ax , $x + ab = acx$
 Transposing, $x - acx = -ab$
 Changing signs, $acx - x = ab$ (Art. 152, Note.)
 Factoring, $(ac - 1)x = ab$
 Dividing by $ac - 1$, $x = \frac{ab}{ac - 1}$

24. Given $x + nx = a$
 Factoring, $(1 + n)x = a$
 Dividing by $1 + n$, $x = \frac{a}{1 + n}$

25. FIRST SOLUTION.

Given $a - cx^{-1} = dx^{-1} - b$
 Multiplying by x , $ax - c = d - bx$
 Transposing, $ax + bx = c + d$
 Factoring, $(a + b)x = c + d$
 Dividing by $a + b$, $x = \frac{c + d}{a + b}$

SECOND SOLUTION.

Given $a - cx^{-1} = dx^{-1} - b$
 Transposing, $-cx^{-1} - dx^{-1} = -a - b$
 Changing signs, $cx^{-1} + dx^{-1} = a + b$
 Factoring, $(c + d)x^{-1} = a + b$
 Dividing by $c + d$, $x^{-1} = \frac{a + b}{c + d}$
 Taking reciprocals, $x = \frac{c + d}{a + b}$

26. Given $\frac{bx}{a} - \frac{d}{c} = \frac{a}{b} - \frac{cx}{d}$
 Mult. by $abcd$, $b^2cdx - abd^2 = a^2cd - abc^2x$
 Transposing, $abc^2x + b^2cdx = a^2cd + abd^2$

Factoring, $b c x (a c + b d) = a d (a c + b d)$

Dividing by $a c + b d$, $b c x = a d$

Dividing by $b c$, $x = \frac{a d}{b c}$

NOTE. If the members of the equation had not been factored and divided by $a c + b d$, we should have obtained

$$x = \frac{a^2 c d + a b d^2}{a b c^2 + b^2 c d} = \frac{a d (a c + b d)}{b c (a c + b d)} = \frac{a d}{b c}.$$

27. Given $\frac{2x+1}{3} = x - \frac{x+3}{4}$

Multiplying by 12, $8x + 4 = 12x - 3x - 9$

Transposing, $8x - 12x + 3x = -9 - 4$

Uniting, $-x = -13$

Changing signs, $x = 13$

29. Given $5 - \frac{x+4}{11} = x - 3$

Multiplying by 11, $55 - x - 4 = 11x - 33$

Transposing, $-x - 11x = -33 + 4 - 55$

Uniting, $-12x = -84$

Dividing by -12 , $x = 7$

30. Given $2x + \frac{24}{4} = \frac{3x}{4} - 4$

Multiplying by 4, $8x + 24 = 3x - 16$

Transposing, $8x - 3x = -16 - 24$

Uniting, $5x = -40$

Dividing by 5, $x = -8$

31. Given $x + \frac{x+8}{4} - \frac{x-6}{3} = x - 2$

Dropping x from both members, $\frac{x+8}{4} - \frac{x-6}{3} = -2$

Multiplying by 12, $3x + 24 - 4x + 24 = -24$

Transposing, $3x - 4x = -24 - 24 - 24$

Uniting, $-x = -72$

Changing signs, $x = 72$

32. Given $\frac{3x}{4} + \frac{x-4}{2} - \frac{4x-40}{8} = x-6$
 Mult. by 8, $6x + 4x - 16 - 4x + 40 = 8x - 48$
 Transposing, $6x + 4x - 4x - 8x = -48 - 40 + 16$
 Uniting, $-2x = -72$
 Dividing by -2 , $x = 36$

33. Given $\frac{5x-11}{4} - \frac{x-1}{10} = \frac{11x-1}{12}$
 Mult. by 60, $75x - 165 - 6x + 6 = 55x - 5$
 Transposing, $75x - 6x - 55x = 165 - 6 - 5$
 Uniting, $14x = 154$
 Dividing by 14, $x = 11$

35. Given $\frac{7x+16}{21} - \frac{x+8}{4x-11} = \frac{x}{3}$
 Mult. by 21, $7x + 16 - \frac{21x+168}{4x-11} = 7x$
 Dropping 7x, $16 - \frac{21x+168}{4x-11} = 0$
 Transposing, $16 = \frac{21x+168}{4x-11}$
 Mult. by $4x-11$, $64x - 176 = 21x + 168$
 Transposing, $64x - 21x = 176 + 168$
 Uniting, $43x = 344$
 Dividing by 43, $x = 8$

NOTE. Thus far, in this Key, the processes of transposing and uniting similar terms have been kept distinct. When there are but two or three similar terms, however, the final result of both operations can be readily obtained *mentally*, and we can thus avoid the labor of writing out the equation which results from transposition alone.

36. Given $\frac{5x-2}{8} + \frac{3x+22}{2(x+3)} = \frac{5x+14}{8}$
 Transposing $\frac{5x-2}{8}$ to the second member, uniting
 it with $\frac{5x+14}{8}$, and reducing,

$$\frac{3x+22}{2(x+3)} = 2$$

5x + 14

5x - 2

and 2 = 2

Mult. by $2x + 6$, $3x + 22 = 4x + 12$

Transposing and uniting, $-x = -10$

Changing signs, $x = 10$

37. Given $\frac{4x+17}{9} - \frac{26x-4}{17x+32} + \frac{2x}{3} = \frac{14x}{12} - \frac{2x-32}{36}$
Change to multiplying by 36 the no many is possible

$$16x + 68 - \frac{936x - 144}{17x + 32} + 24x = 42x - 2x + 32$$

Or, $40x + 68 - \frac{936x - 144}{17x + 32} = 40x + 32.$

Transposing, uniting, and changing signs,

Divide by 36 $\frac{936x - 144}{17x + 32} = 36$

Clearing of fractions, $936x - 144 = 612x + 1152$

Transposing and uniting, $324x = 1296$

Dividing by 324, $x = 4$

38. Given $\frac{13-x}{2} = \frac{6(x-5)}{8} + \frac{8x+15}{6}$

Mult. by 2, and reducing, $13-x = \frac{3(x-5)}{2} + \frac{8x+15}{3}$

Clearing of fractions, $78 - 6x = 9x - 45 + 16x + 30$

Transp. and uniting, $-31x = -93$

Dividing by -31 , $x = 3$

39. Given $\frac{6x+7}{9} + \frac{7x-13}{6x+3} = \frac{2x+4}{3}$

Mult. by 3, $\frac{6x+7}{3} + \frac{7x-13}{2x+1} = 2x+4$

Mult. by 3, $6x+7 + \frac{21x-39}{2x+1} = 6x+12$

Transp. and uniting, $\frac{21x-39}{2x+1} = 5$

Clearing of fractions, $21x - 39 = 10x + 5$

Transp. and uniting, $11x = 44$

Dividing by 11, $x = 4$

40. Given $\frac{2x}{5} + \frac{3x+5}{5x-25} = \frac{6x+13}{15}$
 Mult. by 5, $2x + \frac{3x+5}{x-5} = \frac{6x+13}{3}$
 Mult. by 3, $6x + \frac{9x+15}{x-5} = 6x+13$
 Transp. and uniting, $\frac{9x+15}{x-5} = 13$
 Clearing of fractions, $9x+15 = 13x-65$
 Transp. and uniting, $-4x = -80$
 Dividing by -4 , $x = 20$

41. Given $\frac{x}{1+a} + \frac{x}{1-a} = b$
 Clear. of fract., $x - ax + x + ax = b(1-a^2)$
 Or, $2x = b(1-a^2)$
 Dividing by 2, $x = \frac{b}{2}(1-a^2)$

42. Given $x - \frac{3x-3}{5} + 4 = \frac{20-x}{2} - \frac{6x-8}{7} + \frac{4x-4}{5}$
 Transposing $-\frac{3x-3}{5}$, and uniting it with $\frac{4x-4}{5}$,
 $x + 4 = \frac{20-x}{2} - \frac{6x-8}{7} + \frac{7x-7}{5}$
 Multiplying by 70,
 $70x + 280 = 700 - 35x - 60x + 80 + 98x - 98$
 Transposing,
 $70x + 35x + 60x - 98x = 700 + 80 - 98 - 280$
 Uniting,
 $67x = 402$
 Dividing by 67,
 $x = 6$

43. Given $\frac{ax}{b} - e = \frac{mx}{n} + d$
 Mult. by bn , $anx - ben = bmx + bdn$
 Transposing, $anx - bmx = ben + bdn$
 Dividing by $an - bm$, $x = \frac{ben + bdn}{an - bm}$

44. Given $\frac{3a+x}{x} - 5 - \frac{6}{x} = 0$
 Mult. by x , $3a + x - 5x - 6 = 0$.
 Tr., uniting, and ch. signs, $4x = 3a - 6$
 Dividing by 4, $x = \frac{3a-6}{4}$

45. Given $ax^2 + b^2x = a^2x + bx^2$
 Dividing by x , $ax + b^2 = a^2 + bx$
 Transposing, $ax - bx = a^2 - b^2$
 Dividing by $a - b$, $x = a + b$

46. Given $(a+x)(b+x) - a(b+c) = \frac{a^2c}{b} + x^2$
 Expanding, $ab + ax + bx + x^2 - ab - ac = \frac{a^2c}{b} + x^2$
 Or, $ax + bx - ac = \frac{a^2c}{b}$
 Multiplying by b , $abx + b^2x - abc = a^2c$
 Transposing, $abx + b^2x = a^2c + abc$
 Factoring, $bx(a+b) = ac(a+b)$
 Dividing by $b(a+b)$, $x = \frac{ac}{b}$

NOTE. The value of x is also $\frac{a^2c + abc}{ab + b^2}$, which, reduced to its lowest terms by dividing both numerator and denominator by $a+b$, becomes $\frac{ac}{b}$.

47. Given $\frac{x}{a-b} - \frac{2+x}{a+b} = \frac{c}{a^2-b^2}$
 Mult. by $a^2 - b^2$, $ax + bx - 2a + 2b - ax + bx = c$
 Transp. and uniting, $2bx = c + 2a - 2b$
 Dividing by $2b$, $x = \frac{c + 2a - 2b}{2b}$

48. Given

$$ax - \frac{a^2 - 3bx}{a} - ab^2 = bx + \frac{6bx - 5a^2}{2a} - \frac{bx + 4a}{4}$$

Multiplying by $4a$,

$$4a^2x - 4a^2 + 12bx - 4a^2b^2 = 4abx + 12bx - 10a^2 - abx - 4a^2$$

Cancelling $-4a^2$ and $12bx$,

$$4a^2x - 4a^2b^2 = 4abx - 10a^2 - abx$$

Uniting similar terms, and dividing each term by a ,

$$4ax - 4ab^2 = 3bx - 10a$$

Transposing, $4ax - 3bx = 4ab^2 - 10a$

$$\text{Dividing by } 4a - 3b, \quad x = \frac{4ab^2 - 10a}{4a - 3b}$$

$$\text{Or,} \quad x = \frac{2a(2b^2 - 5)}{4a - 3b}$$

PROBLEMS

LEADING TO SIMPLE EQUATIONS CONTAINING ONE UNKNOWN QUANTITY.

(ART. 167, pp. 120-137.)

2. Let $x =$ the smaller number,
and $x + 16 =$ the larger number.

$$\text{Their sum,} \quad x + x + 16 = 40$$

$$\text{Transp. and uniting,} \quad 2x = 24$$

$$\text{Dividing by 2,} \quad x = 12, \text{ the smaller number.}$$

$$\text{Then,} \quad x + 16 = 28, \text{ the larger number.}$$

3. Let $x =$ no. votes for unsuccessful candidate,
and $x + 120 =$ no. votes for successful candidate.

$$\text{Their sum,} \quad x + x + 120 = 1296$$

$$\text{Transp. and uniting,} \quad 2x = 1176$$

$$\text{Dividing by 2,} \quad x = 588, \text{ no. votes for one.}$$

$$\text{Then,} \quad x + 120 = 708, \text{ no. votes for the other.}$$

4. Let $x =$ smaller number,
 and $x + 13 =$ larger number.

 Their sum and 17, $x + x + 13 + 17 = 62$
 Transposing and uniting, $2x = 32$
 Dividing by 2, $x = 16$, one number.
 Then, $x + 13 = 29$, other number.

5. Let $x =$ sum due A,
 $2x =$ sum due B,
 and $3x =$ sum due C.

 The whole sum, $6x = 3000$
 Dividing by 6, $x = 500$, sum due A.
 Then, $2x = 1000$, sum due B.
 Also, $3x = 1500$, sum due C.

6. Let $x =$ no. of children,
 $2x =$ no. of women,
 and $4x =$ no. of men.

 The whole number, $7x = 266$
 Dividing by 7, $x = 38$, no. children.
 Then $2x = 76$, no. women.
 Also, $4x = 152$, no. men.

8. Let $x =$ number of days required.
 Then $36x =$ distance traveled by one,
 and $30x =$ dist. traveled by the other.

 Their sum, $36x + 30x = 396$
 Uniting terms, $66x = 396$
 Dividing by 66, $x = 6$, no. days.
 Then, $36x = 216$, no. miles one trav.
 Also, $30x = 180$, no. miles other trav.

9. Let $x =$ no. hours horseman rides,
 and $x + 10 =$ no. hours first person trav.
 Then $9x =$ distance horseman rides,
 and $4(x + 10) =$ dist. first person travels.
-
- Hence, $9x = 4x + 40$
 Transp. and uniting, $5x = 40$
 Dividing by 5, $x = 8$, no. hours horseman rides.
10. Let $x =$ price of the house,
 and $850 - x =$ price of the garden.
-
- Then, $5x = 12(850 - x)$
 Or, $5x = 10200 - 12x$
 Transp. and uniting, $17x = 10200$
 Dividing by 17, $x = 600$, price of house.
 Then, $850 - x = 250$, price of garden.
11. Let $x =$ value of a sheep.
 Then $72x + 35 =$ value of A's share,
 and $92x - 35 =$ value of B's share.
-
- Hence, $92x - 35 = 72x + 35$
 Transp. and uniting, $20x = 70$
 Dividing by 20, $x = 3\frac{1}{2}$, value of a sheep.
13. Let $x =$ James's age,
 and $\frac{8x}{5} =$ John's age.
-
- The sum, $x + \frac{8x}{5} = 39$
 Clear. of fractions, $5x + 8x = 195$
 Uniting terms, $13x = 195$
 Whence, $x = 15$, James's age.
 Then, $\frac{8x}{5} = 24$, John's age.

14. Let $x = \text{C's share,}$
 $\frac{2x}{3} = \text{A's share,}$
 and $\frac{3x}{4} = \text{B's share.}$
-
- Their sum, $x + \frac{2x}{3} + \frac{3x}{4} = 145$
 Clear. of fract., $12x + 8x + 9x = 1740$
 Uniting terms, $29x = 1740$
 Whence, $x = 60, \text{C's share.}$
 Then, $\frac{2x}{3} = 40, \text{A's share.}$
 Also, $\frac{3x}{4} = 45, \text{B's share.}$

15. Let $x = \text{his age.}$
 Then, $x + \frac{x}{2} + \frac{x}{3} = 44$
 Clear. of fract., $6x + 3x + 2x = 264$
 Or, $11x = 264$
 Whence, $x = 24, \text{his age.}$

17. Let $6x = \text{length of the pole.}$
 Then $\frac{6x}{2} = 3x = \text{half of its length,}$
 and $\frac{6x}{3} = 2x = \text{one third of its length.}$

Hence, $3x + 2x + 4 = 6x$
 Or, $-x = -4$
 Whence, $x = 4$
 Then, $6x = 24, \text{length of the pole.}$

18. Let $35x = \text{whole journey,}$
 $14x = \frac{2}{3} \text{ of journey,}$
 and $15x = \frac{1}{3} \text{ of journey.}$

Then, $14x + 15x + 30 = 35x$

- Or, $-6x = -30$
 Whence, $x = 5$
 Then, $35x = 175$, the whole journey.
19. Let $6x =$ capacity of the cask,
 $2x =$ amount of oil,
 and $x = \frac{1}{2}$ of the oil.
 Then, $2x - 21 = x$
 Or, $x = 21$
 Hence, $6x = 126$, capacity of cask.
20. Let $x =$ age of youngest,
 $x + 2 =$ age of next,
 and $x + 4 =$ age of oldest.
 The sum, $3x + 6 = 24$
 Transp. and uniting, $3x = 18$
 Whence, $x = 6$, age of youngest.
 Then, $x + 2 = 8$, age of next.
 Also, $x + 4 = 10$, age of oldest.
22. Let $x =$ no. lbs. of each.
 Then, $ax + bx + cx = d$
 Whence, $x = \frac{d}{a + b + c}$, no. lbs. of each.
23. Let $x =$ my age.
 Then, $2x + b = a$
 Transposing, $2x = a - b$
 Whence, $x = \frac{a - b}{2}$, my age.
24. Let $x =$ no. votes for successful candidate,
 and $x - b =$ no. votes for other candidates.
 The sum, $x + x - b = a$
 Or, $2x = a + b$
 Whence, $x = \frac{a + b}{2}$, no. votes for suc. cand.

25. Let $x = \text{value of horse,}$
 and $\frac{3x}{2} = \text{value of carriage.}$
-
- Their sum, $x + \frac{3x}{2} = c$
 Clear. of fractions, $2x + 3x = 2c$
 Or, $5x = 2c$
 Whence, $x = \frac{2c}{5}, \text{ value of horse.}$
 Then, $\frac{3x}{2} = \frac{3c}{5}, \text{ val. of carriage.}$

26. Let $x = \text{no. days second travels,}$
 and $x + n = \text{no. days first travels.}$
 Then $bx = \text{distance second travels,}$
 and $a(x + n) = \text{distance first travels.}$
-
- Therefore, $bx = a(x + n)$
 Or, $bx = ax + na$
 Transposing, $bx - ax = na$
 Whence, $x = \frac{na}{b-a}, \text{ no. days required.}$

27. Let $x = \text{the number.}$
- Then, $\frac{7x+3}{2} - 4 = 15$
 Clear. of fractions, $7x + 3 - 8 = 30$
 Transposing and uniting, $7x = 35$
 Whence, $x = 5, \text{ the number.}$

28. Let $x = \text{length of rod.}$
- Then, $x - \frac{x}{5} - \left(\frac{x}{6} + 3\right) = 16$
 Mult. by 30, $30x - 6x - 5x - 90 = 480$
 Transp. and uniting, $19x = 570$
 Whence, $x = 30, \text{ length rod.}$

30. Let $x =$ price of first house.
 Then $x + \frac{x}{2} = \frac{3x}{2} =$ price of second house,
 $\frac{3x}{2} + \frac{3x}{4} = \frac{9x}{4} =$ price of third house,
 and $x + \frac{9x}{4} = \frac{13x}{4} =$ price of fourth house.

The sum, $x + \frac{3x}{2} + \frac{9x}{4} + \frac{13x}{4} = 8000$

Clearing of fractions,

$$4x + 6x + 9x + 13x = 32000$$

Uniting, $32x = 32000$

Whence, $x = 1000$, price 1st house.

Then, $\frac{3x}{2} = 1500$, price 2d house.

Also, $\frac{9x}{4} = 2250$, price 3d house.

And, $\frac{13x}{4} = 3250$, price 4th house.

NOTE. Fractions may be avoided by using $4x$ for the price of the first house, and, consequently, $6x$, $9x$, and $13x$ for the prices of the others. The value of x will then be found to be 250.

31. Let $x =$ son's age,
 and $3x =$ father's age.
-
- Then, $3x - 5 = 4(x - 5)$
 Or, $3x - 5 = 4x - 20$
 Transp. and uniting, $-x = -15$
 Changing signs, $x = 15$, son's age.
 Then, $3x = 45$, father's age.

32. Let $x =$ time of first,
 and $10 - x =$ time of second.
 Then $14x =$ am't deliv. by first,
 and $9(10 - x) =$ am't deliv. by second.

The sum, $14x + 90 - 9x = 120$

$$\begin{array}{ll}
 \text{Transp. and uniting,} & 5x = 30 \\
 \text{Whence,} & x = 6, \text{ time of first.} \\
 \text{Then,} & 10 - x = 4, \text{ time of second.}
 \end{array}$$

34. The answer to this problem may be obtained either by substitution in the general formula found by Prob. 33, by a statement similar to that of Problem 33, or as follows :

$$\begin{array}{ll}
 \text{Let} & x = \text{the time required.} \\
 \text{Then} & \frac{1}{x} = \text{what both can do in one day.} \\
 \text{Also} & \frac{1}{3} = \text{what A can do in one day,} \\
 \text{and} & \frac{1}{7} = \text{what B can do in one day.}
 \end{array}$$

$$\text{Then,} \quad \frac{1}{3} + \frac{1}{7} = \frac{1}{x}$$

$$\text{Clear. of fract., } 7x + 3x = 21$$

$$\text{Or,} \quad 10x = 21$$

$$\text{Whence,} \quad x = 2\frac{1}{10}, \text{ time required.}$$

35. The fractional part of the whole work performed by A, B, and C, respectively, in one day will be represented by $\frac{1}{a}$, $\frac{1}{b}$, and $\frac{1}{c}$.

$$\begin{array}{ll}
 \text{Let} & x = \text{the time required.} \\
 \text{Then} & \frac{1}{x} = \text{what all can do in one day.}
 \end{array}$$

$$\text{Hence,} \quad \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{x}$$

$$\text{Clear. of fractions, } bcx + acx + abx = abc$$

$$\text{Whence,} \quad x = \frac{abc}{bc + ac + ab}$$

$$\text{Or,} \quad x = \frac{abc}{ab + ac + bc}, \text{ time req.}$$

NOTE. Since each would work x days, the parts performed by each would be $\frac{x}{a}$, $\frac{x}{b}$, and $\frac{x}{c}$; and as they would perform the whole work in that time, the equation might be stated thus :

$$\frac{x}{a} + \frac{x}{b} + \frac{x}{c} = 1.$$

36. Substituting 2 for a , 3 for b , and 4 for c , in the formula just obtained, we have

$$x = \frac{a b c}{a b + a c + b c} = \frac{24}{6 + 8 + 12} = \frac{24}{26} = \frac{12}{13}$$

$$\frac{12}{13} \text{ of } 60 \text{ m.} = 55\frac{5}{13} \text{ m.} = 55 \text{ m. } 23\frac{1}{13} \text{ s.}$$

This problem may also be solved independently of the preceding formula, thus :

Let x = the time required.

Then,
$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{1}{x}$$

Clearing of fractions, $6x + 4x + 3x = 12$

Or, $13x = 12$

Whence, $x = \frac{12}{13}$, the time required.

$$\frac{12}{13} \text{ h.} = 55 \text{ m. } 23\frac{1}{13} \text{ sec.}$$

38. Let x = no. bushels of rye,
and $x + 50$ = no. bushels of the mixture.
Then $\frac{9x}{2}$ = value of x bush. rye at $4\frac{1}{2}$ s.,
and $5(x + 50)$ = value of the mixture.
Also $50 \times 6 = 300$ = value of the wheat.

Then, $5x + 250 = \frac{9x}{2} + 300$

Clearing of fractions, $10x + 500 = 9x + 600$

Transp. and uniting, $x = 100$, no. bush. rye.

39. Let x = no. gals. water,
and $x + 40$ = no. gals. mixture.

Then, $(x + 40) \times 4.50 = 40 \times 7$

$$4.5x + 180 = 280$$

Or,
$$\frac{9x + 360}{2} = 280$$

Clearing of fractions, $9x + 360 = 560$

Transp. and uniting, $9x = 200$

Whence, $x = 22\frac{2}{3}$, no. gals. water.

41. Let $3x =$ his money.
 Then $2x =$ amount at 6 per cent.
 and $x =$ amount at 8 per cent.

$$2x \times \frac{6}{100} = \frac{12x}{100} = \text{income at 6 per cent.}$$

$$x \times \frac{8}{100} = \frac{8x}{100} = \text{income at 8 per cent.}$$

Then,
$$\frac{12x}{100} + \frac{8x}{100} = 1200$$

Clearing of fractions, $12x + 8x = 120000$

Or, $20x = 120000$

Whence, $x = 6000$, am't at 8 p. c.

Then, $2x = 12000$, am't at 6 p. c.

42. Let $x =$ rent last year,
 and $x + \frac{8x}{100} =$ rent this year.

Then,
$$x + \frac{8x}{100} = 1890$$

Clearing of fractions, $100x + 8x = 189000$

Or, $108x = 189000$

Whence, $x = 1750$, rent last year.

43. Let $x =$ the original capital.

Then $x + \frac{40x}{100} - 3000 = \frac{7x}{5} - 3000 =$ capital at the
 commencement of the second year,

and $\frac{7x}{5} - 3000 + \frac{40}{100} \left(\frac{7x}{5} - 3000 \right) - 3000$
 $= \frac{7}{5} \left(\frac{7x}{5} - 3000 \right) - 3000 =$ capital at the end of the
 second year.

$$\text{Hence, } \frac{7}{5} \left(\frac{7x}{5} - 3000 \right) - 3000 = x + \frac{60x}{100}$$

$$\text{Or, } \frac{49x}{25} - \frac{21000}{5} - 3000 = \frac{8x}{5}$$

$$\text{Clear. of fractions, } 49x - 105000 - 75000 = 40x$$

$$\text{Transp. and uniting, } 9x = 180000$$

$$\text{Whence, } x = 20000, \text{ the capital.}$$

NOTE. The letter a may be used in place of 3000, and we shall obtain $\frac{7x}{5} - a$ and $\frac{7}{5} \left(\frac{7x}{5} - a \right) - a$ as expressions for the capital at the times above specified. The equation becomes

$$\frac{49x}{25} - \frac{7a}{5} - a = \frac{8x}{5},$$

$$\text{which gives } x = \frac{20a}{8} = 20 \times 1000.$$

$$45. \quad \text{Let } x = \text{the amount.}$$

$$\text{Then, } \frac{3x}{7} + \frac{x}{8} = 930$$

$$\text{Clearing of fractions, } 24x + 7x = 930 \times 56$$

$$\text{Or, } 31x = 930 \times 56$$

$$\text{Whence, } x = 30 \times 56$$

$$\text{Or, } x = 1680, \text{ the am't req.}$$

$$47. \quad \begin{array}{ll} \text{Let} & x = \text{the greater part,} \\ \text{and} & 34 - x = \text{the less part.} \end{array}$$

$$\text{Then, } x - 18 : 18 = (34 - x) : 2 : 3$$

$$\text{Or, } x - 18 : x - 16 :: 2 : 3$$

Converting the proportion to an equation (Art. 166),

$$3(x - 18) = 2(x - 16)$$

$$\text{Or, } 3x - 54 = 2x - 32$$

$$\text{Transposing and uniting, } x = 22, \text{ the greater part.}$$

$$\text{Then, } 34 - x = 12, \text{ the less part.}$$

$$48. \quad \text{FIRST SOLUTION.}$$

$$\text{Let } x = \text{no. dollar pieces,}$$

$$\text{and } 264 - x = \text{no. eagles.}$$

$$\text{Then, } x : 264 - x :: 9 : 2$$

Mult. ext. and means, $2x = 2376 - 9x$
 Transp. and uniting, $11x = 2376$
 Whence, $x = 216$, no. dollar pieces.
 Then, $264 - x = 48$, no. eagles.

SECOND SOLUTION.

Let $9x =$ the number of dollar pieces,
 and $2x =$ the number of eagles.
 Then, $9x + 2x = 264$
 Or, $11x = 264$
 Whence, $x = 24$
 Then, $2x = 48$, no. eagles.
 Also, $9x = 216$, no. dollar pieces.

49. Let $3x =$ age of one,
 and $4x =$ age of the other.
 Then, $3x - 5 : 4x - 5 :: 2 : 3$
 Mult. extremes and means, $9x - 15 = 8x - 10$
 Transposing and uniting, $x = 5$
 Whence, $3x = 15$, age of one.
 Also, $4x = 20$, age of other.

50. FIRST SOLUTION.

Let $x =$ the price per yard.
 Then $\frac{5}{x} =$ length of first piece,
 and $\frac{6.50}{x} = \frac{13}{2x} =$ length second piece.

From the conditions, $\frac{5}{x} + 10 : \frac{13}{2x} + 10 :: 5 : 6$

Mult. ext. and means, $\frac{30}{x} + 60 = \frac{65}{2x} + 50$

Clear. of fractions, $60 + 120x = 65 + 100x$

Transposing and uniting, $20x = 5$

Whence, $x = \frac{1}{4}$

Then, $\frac{5}{x} = 20$, length of first.

Also, $\frac{13}{2x} = 26$, length of second.

SECOND SOLUTION.

As the price of each per yard is the same, it follows that the lengths of the two pieces must be in the ratio of their full prices; that is, as 5 to 6.50, or 10 to 13.

Let $10x =$ length of first piece,
and $13x =$ length second piece.

Then, $10x + 10 : 13x + 10 :: 5 : 6$

Mult. ex. and means, $60x + 60 = 65x + 50$

Transp. and uniting, $-5x = -10$

Whence, $x = 2$

Then, $10x = 20$, length of first.

Also, $13x = 26$, length of second.

NOTE. This problem furnishes a good opportunity for the use of negative exponents. If x represent the price per yard, then $5x^{-1}$ and $\frac{13}{2}x^{-1}$ will represent the lengths of each, and

$$5x^{-1} + 10 : \frac{13}{2}x^{-1} + 10 :: 5 : 6,$$

from which $x^{-1} = 4$, $5x^{-1} = 20$, and $\frac{13}{2}x^{-1} = 26$.

52. Let $x =$ A's capital,
and $6300 - x =$ B's capital.

Then, $12x = 8(6300 - x)$

Or, $12x = 50400 - 8x$

Transposing and uniting, $20x = 50400$

Whence, $x = 2520$, A's capital.

Then, $6300 - x = 3780$, B's capital.

54. Let $x =$ no. days he worked,
and $48 - x =$ no. days he was idle.
Then $2x =$ am't rec'd for labor,
and $48 - x =$ am't deducted for board.

From the conditions, $2x - (48 - x) = 42$

Or, $2x - 48 + x = 42$

Transposing and uniting, $3x = 90$

Whence, $x = 30$, no. days he worked.

55. Let $x =$ no. quarts in each.
 Then $x - 34 =$ no. quarts left in first,
 and $x - 80 =$ no. qts. left in second.

From the conditions, $x - 34 = 2(x - 80)$

Or, $x - 34 = 2x - 160$

Transp. and uniting, $-x = -126$

Or, $x = 126$, no. quarts in each.

56. Let $x =$ no. persons.
 Then $\frac{2x}{3} \times 18 = 12x =$ am't rec'd by two thirds,
 and $\frac{x}{3} \times 30 = 10x =$ am't rec'd by one third.

Therefore, $12x + 10x = 660$

Or, $22x = 660$

Whence, $x = 30$, no. persons.

57. Let $x =$ weight of the body.
 Then $12 + \frac{x}{2} =$ weight of the tail,
 and $24 + \frac{3x}{2} =$ weight of the fish.

From the conditions, $x = 12 + 12 + \frac{x}{2} + 26$

Or, $x = 50 + \frac{x}{2}$

Clearing of fractions, $2x = 100 + x$

Whence, $x = 100$, weight of body.

Then, $24 + \frac{3x}{2} = 174$, weight of fish.

58. Let $x =$ rate river flows.
 Then $9 + x =$ rate he moves down,
 and $9 - x =$ rate he moves up.

$$\text{Therefore, } 9 + x = 2(9 - x)$$

$$\text{Or, } 9 + x = 18 - 2x$$

$$\text{Transposing and uniting, } 3x = 9$$

$$\text{Whence, } x = 3, \text{ rate river flows.}$$

60. Let $x =$ number of hogs,
 and $35 - x =$ number of pigs.

$$\text{Then, } 1250x + 250(35 - x) = 19750$$

$$\text{Dividing by 250, } 5x + 35 - x = 79$$

$$\text{Transposing and uniting, } 4x = 44$$

$$\text{Whence, } x = 11, \text{ no. hogs.}$$

$$\text{Then, } 35 - x = 24, \text{ no. pigs.}$$

61. Let $x =$ her age.
 Then $\frac{x}{4} =$ half of the remainder.

$$\text{Therefore, } x - \frac{x}{2} - \frac{x}{4} = 19$$

$$\text{Clearing of fractions, } 4x - 2x - x = 76$$

$$\text{Uniting, } x = 76, \text{ her age.}$$

62. Let $x =$ price of the clothes.
 Then $\frac{90 + x}{12} =$ monthly wages.

$$\text{Therefore, } 7\left(\frac{90 + x}{12}\right) = 33\frac{1}{2} + x$$

$$\text{Or, } \frac{630 + 7x}{12} = \frac{135}{4} + x$$

$$\text{Clearing of fractions, } 630 + 7x = 405 + 12x$$

$$\text{Transposing and uniting, } -5x = -225$$

$$\text{Whence, } x = 45, \text{ price of clothes.}$$

$$630 + 7x = 135 + 12x$$

$$630 - 135 = 12x - 7x$$

63. Let $x = \text{cost of the house,}$
 and $a = 12000 = \text{whole am't of money.}$
 Then $\frac{a-x}{3} \times \frac{4}{100} = \text{income at 4 per cent,}$
 and $\frac{2a-2x}{3} \times \frac{5}{100} = \text{income at 5 per cent.}$

Therefore, $\frac{4a-4x}{300} + \frac{10a-10x}{300} = 392$

Clearing of fractions, $4a-4x + 10a-10x = 117600$

Transposing and uniting, $14a-14x = 117600$

Dividing by 14, $a-x = 8400$

Whence, $x = a - 8400 = 12000 - 8400$

Or, $x = 3600, \text{ cost of the house.}$

64. Let $x = \text{no. children,}$
 and $3x-8 = \text{amount of money.}$

Therefore, $3x-8 = 2x+3$

Transposing and uniting, $x = 11, \text{ no. children.}$

65. Let $20x = \text{A's sum,}$
 $12x = \text{B's sum,}$
 and $18x = \text{C's sum.}$

Then, $20x + 12x + 18x = 11800$

Uniting, $50x = 11800$

Whence, $x = 236$

Then, $20x = 4720, \text{ A's sum.}$

Also, $12x = 2832, \text{ B's sum,}$

and $18x = 4248, \text{ C's sum.}$

66. Let $x = \text{his money at first.}$

Then $2x-16 = \text{money at close of first eve,}$

$4x-32-16 = 4x-48 = \text{money at close of second eve,}$

$8x-96-16 = 8x-112 = \text{money at close of third eve,}$

$16x-224-16 = 16x-240 = \text{money at close of fourth eve.}$

Therefore, $16x-240 = 0$

Or, $16x = 240$
 Whence, $x = 15$, his money at first.

NOTE. The preceding solution may be abbreviated by the use of a for 16, when the different quantities become $2x - a$, $4x - 3a$, $8x - 7a$, and $16x - 15a$; and the equation becomes

$$16x = 15a, \text{ or } ax = 15a,$$

whence $x = 15$.

67. Let $x =$ the number.

Then, $x + \frac{x}{4} + \frac{x}{5} + \frac{x}{8} = 126$

Clearing of fractions,

$$40x + 10x + 8x + 5x = 5040$$

Uniting, $63x = 5040$

Whence, $x = 80$, the number.

68. Let $x =$ whole no. dollars.

Then $2 + \frac{x-2}{6} = \frac{10+x}{6} =$ no. taken by A,

$$x - \frac{10+x}{6} = \frac{5x-10}{6} =$$
 no. left by A,

and $3 + \frac{1}{6} \left(\frac{5x-10}{6} - 3 \right) = \frac{80+5x}{36} =$ no. taken by B.

Therefore, $\frac{10+x}{6} = \frac{80+5x}{36}$

Clear. of fractions, $60 + 6x = 80 + 5x$

Transp. and uniting, $x = 20$, whole no. dollars.

Then, $\frac{10+x}{6} = 5$, no. taken by each.

69. Let $x =$ the number.

Then, $\frac{3x-8}{2} = x - 2$

Clearing of fractions, $3x - 8 = 2x - 4$

Transposing and uniting, $x = 4$, the number.

Ans. 4.

70. Let $x =$ no. days required.
 Then $\frac{1}{x} =$ part cons. by man in one day.
 Also $\frac{1}{30} =$ part cons. by wife in one day,
 and $\frac{1}{12} =$ part cons. by both in one day.

$$\text{Therefore,} \qquad \frac{1}{12} = \frac{1}{x} + \frac{1}{30}$$

$$\text{Clearing of fractions,} \qquad 5x = 60 + 2x$$

$$\text{Transposing and uniting,} \quad 3x = 60$$

$$\text{Whence,} \qquad x = 20, \text{ no. days required.}$$

71. FIRST SOLUTION.

Let x represent the number of hour spaces passed over by the hour-hand from 12 o'clock until the time when the hands will next be together. Then $12x$ will represent the number of hour spaces passed over by the minute-hand during the same time, for the latter moves 12 times as fast as the former.

In order that the two may be together, the minute-hand must first move once around the whole dial, or 12 hour spaces, and then over the same space that the hour-hand has passed over.

$$\text{Hence,} \qquad 12x = 12 + x$$

$$\text{Transp. and uniting,} \quad 11x = 12$$

$$\text{Whence,} \qquad x = 1 \frac{1}{11}, \text{ no. hours req.}$$

$$1 \frac{1}{11} \text{ h.} = 1 \text{ h. } 5 \frac{5}{11} \text{ m.}$$

NOTE. If x represent the number of minute spaces passed over by the minute-hand, then the equation becomes

$$x = 60 + \frac{x}{12}, \text{ or } 12x = 720 + x;$$

$$\text{whence,} \qquad 11x = 720, \qquad \text{and } x = 65 \frac{5}{11} \text{ minutes.}$$

SECOND SOLUTION.

As it is evident that the hands cannot be together until after the minute-hand has again passed the 12

mark, let x represent the number of minute spaces passed over by the minute-hand after 1 o'clock, or its distance from the 12 mark at the required time. At 1 o'clock the hour-hand is 5 minute spaces from 12, and while the minute-hand moves over x spaces, the hour-hand must pass over $\frac{x}{12}$; hence $5 + \frac{x}{12}$ will also represent the distance of the hour-hand from the 12 mark at the required time.

$$\begin{array}{ll} \text{Then,} & x = 5 + \frac{x}{12} \\ \text{Clearing of fractions,} & 12x = 60 + x \\ \text{Transp. and uniting,} & 11x = 60 \\ \text{Whence,} & x = 5\frac{5}{11}, \text{ no. min. after 1.} \end{array}$$

72. If the number in the first flock increased by 2 is equal to the number in the second diminished by 2, then the second flock numbers 4 more than the first. Again, if the number in the first increased by 2 is equal to the number in the third multiplied by 2, then the third numbers one half as many as the first increased by 2. Further, if the number in the third multiplied by 2 is equal to the number in the fourth divided by 2, then the fourth numbers four times as many as the third.

$$\begin{array}{ll} \text{Let} & x = \text{no. in the first.} \\ \text{Then} & x + 4 = \text{no. in the second,} \\ & \frac{x + 2}{2} = \text{no. in the third,} \\ \text{and} & 2x + 4 = \text{no. in the fourth.} \end{array}$$

$$\text{Therefore, } x + x + 4 + \frac{x + 2}{2} + 2x + 4 = 90$$

$$\text{Or, } 4x + 8 + \frac{x + 2}{2} = 90$$

$$\text{Clearing of fractions, } 8x + 16 + x + 2 = 180$$

$$\text{Transposing and uniting, } 9x = 162$$

Whence, $x = 18$, no. in first.
 Then, $x + 4 = 22$, no. in second.
 Also, $\frac{x + 2}{2} = 10$, no. in third.
 And, $2x + 4 = 40$, no. in fourth.

73. Let $x =$ duty per lb. after reduction,
 and $m =$ no. lbs. originally consumed.
 Then $\frac{3m}{2} =$ no. lbs. afterwards consumed,
 $6m =$ original revenue,
 and $\frac{3mx}{2} =$ revenue after reduction.
 Therefore, $\frac{3mx}{2} = 6m - \frac{6m}{3}$
 Or, $\frac{3mx}{2} = 4m$
 Dividing by m , $\frac{3x}{2} = 4$
 Multiplying by 2, $3x = 8$
 Whence, $x = 2\frac{2}{3}$, duty per lb. after reduction.

NOTE. The above equation may be obtained without the use of m , as follows: If the amount consumed had remained the same, the revenue's falling one third would have been caused by reducing the duty one third, that is, to 4 cents a pound. Again, the increase of one half in consumption had the same effect upon the revenue as an increase of one half in the duty on a pound. Hence, $x + \frac{x}{2} = 4$.

74. Let $x =$ no. greyhound's leaps.
 Then $\frac{3x}{2} =$ whole no. hare's leaps,
 and $\frac{4x}{3} =$ hare's leaps aft. grey. starts.
 Therefore, $\frac{3x}{2} = \frac{4x}{3} + 50$
 Clear. of fractions, $9x = 8x + 300$
 Transp. and uniting, $x = 300$, no. greyhound's leaps.

75. Let $x =$ no. men on a side at first.
 Then $x^2 + 21 =$ the whole no of men.
- Therefore, $(x + 1)^2 = x^2 + 21 + 200$
 Or, $x^2 + 2x + 1 = x^2 + 221$
 Whence, $2x = 220$
 Dividing by 2, $x = 110$
 Squaring, $x^2 = 12100$
 Then, $x^2 + 21 = 12121$, whole no. men.

SIMPLE EQUATIONS CONTAINING TWO UNKNOWN QUANTITIES.

(ART. 172, pp. 138 – 140.)

$$\begin{array}{ll} 2. \text{ Given} & \begin{cases} 2x + 3y = 23 & (1) \\ 5x - 2y = 10 & (2) \end{cases} \end{array}$$

$$\text{Transp. } 3y \text{ in (1) and div. by 2,} \quad x = \frac{23 - 3y}{2} \quad (3)$$

$$\text{Transp. } 2y \text{ in (2) and div. by 5,} \quad x = \frac{10 + 2y}{5} \quad (4)$$

$$\text{By (3), (4), and Ax. 7,} \quad \frac{10 + 2y}{5} = \frac{23 - 3y}{2} \quad (5)$$

$$\text{Clearing of fractions,} \quad 20 + 4y = 115 - 15y \quad (6)$$

$$\text{Transposing and uniting,} \quad 19y = 95 \quad (7)$$

$$\text{Dividing by 19,} \quad y = 5 \quad (8)$$

$$\text{Substituting in (4),} \quad x = \frac{10 + 10}{5} = 4 \quad (9)$$

$$\begin{array}{ll} 3. \text{ Given} & \begin{cases} 4x + y = 34 & (1) \\ x + 4y = 16 & (2) \end{cases} \end{array}$$

$$\text{From (1),} \quad x = \frac{34 - y}{4} \quad (3)$$

$$\text{From (2),} \quad x = 16 - 4y \quad (4)$$

By Ax. 7, $\frac{34 - y}{4} = 16 - 4y$ (5)

Clearing of fractions, $34 - y = 64 - 16y$ (6)

Transposing and uniting, $15y = 30$ (7)

Whence, $y = 2$ (8)

Substituting in (4), $x = 16 - 8 = 8$ (9)

4. Given $\begin{cases} 5x - 3y = 9 & (1) \\ 2x + 5y = 16 & (2) \end{cases}$

From (1), $x = \frac{9 + 3y}{5}$ (3)

From (2), $x = \frac{16 - 5y}{2}$ (4)

By Ax. 7, $\frac{9 + 3y}{5} = \frac{16 - 5y}{2}$ (5)

Clearing of fractions, $18 + 6y = 80 - 25y$ (6)

Transposing and uniting, $31y = 62$ (7)

Whence, $y = 2$ (8)

Substituting in (3), $x = \frac{9 + 6}{5} = 3$ (9)

5. Given $\begin{cases} 7x + 3y = 13 & (1) \\ 5x + 2y = 9 & (2) \end{cases}$

From (1), $y = \frac{13 - 7x}{3}$ (3)

From (2), $y = \frac{9 - 5x}{2}$ (4)

By Ax. 7, $\frac{9 - 5x}{2} = \frac{13 - 7x}{3}$ (5)

Clearing of fractions, $27 - 15x = 26 - 14x$ (6)

Transposing and uniting, $-x = -1$

Or, $x = 1$

Substituting in (4), $y = \frac{9 - 5}{2} = 2$ } Ans.

$$\begin{array}{ll} 6. \text{ Given} & \begin{cases} 8x - 7y = -15 & (1) \\ 3y - 6x = -9 & (2) \end{cases} \end{array}$$

$$\text{From (1),} \quad x = \frac{7y - 15}{8} \quad (3)$$

$$\text{From (2),} \quad x = \frac{3y + 9}{6} \quad (4)$$

$$\text{By Ax. 7,} \quad \frac{7y - 15}{8} = \frac{3y + 9}{6} \quad (5)$$

$$\text{Multiplying by 24,} \quad 21y - 45 = 12y + 36 \quad (6)$$

$$\text{Transposing and uniting,} \quad 9y = 81 \quad (7)$$

$$\text{Whence,} \quad y = 9 \quad (8)$$

$$\text{Substituting in (4),} \quad x = \frac{27 + 9}{6} = 6 \quad (9)$$

$$\begin{array}{ll} 7. \text{ Given} & \begin{cases} 14x + 6y = 0 & (1) \\ 6x - 46 = 4y & (2) \end{cases} \end{array}$$

$$\text{From (1),} \quad y = -\frac{14x}{6} = -\frac{7x}{3} \quad (3)$$

$$\text{From (2),} \quad y = \frac{6x - 46}{4} = \frac{3x - 23}{2} \quad (4)$$

$$\text{By Ax. 7,} \quad \frac{3x - 23}{2} = -\frac{7x}{3} \quad (5)$$

$$\text{Clearing of fractions,} \quad 9x - 69 = -14x \quad (6)$$

$$\text{Transposing and uniting,} \quad 23x = 69 \quad (7)$$

$$\text{Whence,} \quad x = 3 \quad (8)$$

$$\text{Substituting in (3),} \quad y = -\frac{7 \times 3}{3} = -7 \quad (9)$$

$$\begin{array}{ll} 8. \text{ Given} & \begin{cases} \frac{x}{2} + \frac{y}{3} = 7 & (1) \\ \frac{x}{3} + \frac{y}{2} = 8 & (2) \end{cases} \end{array}$$

$$\text{Clearing (1) of fractions,} \quad 3x + 2y = 42 \quad (3)$$

$$\text{Clearing (2) of fractions,} \quad 2x + 3y = 48 \quad (4)$$

$$\text{From (3),} \quad x = \frac{42 - 2y}{3} \quad (5)$$

$$\text{From (4),} \quad x = \frac{48 - 3y}{2} \quad (6)$$

$$\text{By Ax. 7,} \quad \frac{42 - 2y}{3} = \frac{48 - 3y}{2} \quad (7)$$

$$\text{Clearing of fractions,} \quad 84 - 4y = 144 - 9y \quad (8)$$

$$\text{Transposing and uniting,} \quad 5y = 60 \quad (9)$$

$$\text{Whence,} \quad y = 12 \quad (10)$$

$$\text{Substituting in (5),} \quad x = \frac{42 - 24}{3} = 6 \quad (11)$$

$$\begin{array}{l} 9. \text{ Given} \\ \left\{ \begin{array}{l} x + 2y = 17 \\ 3x - y = 2 \end{array} \right. \end{array} \quad \begin{array}{l} (1) \\ (2) \end{array}$$

$$\text{From (1),} \quad x = 17 - 2y \quad (3)$$

$$\text{From (2),} \quad x = \frac{2 + y}{3} \quad (4)$$

$$\text{By Ax. 7,} \quad 17 - 2y = \frac{2 + y}{3} \quad (5)$$

$$\text{Clearing of fractions,} \quad 51 - 6y = 2 + y \quad (6)$$

$$\text{Transposing and uniting,} \quad -7y = -49 \quad (7)$$

$$\text{Whence,} \quad y = 7 \quad (8)$$

$$\text{Substituting in (3),} \quad x = 17 - 14 = 3 \quad (9)$$

$$\begin{array}{l} 10. \text{ Given} \\ \left\{ \begin{array}{l} \frac{x}{2} - y = 1 \\ x - \frac{y}{2} = 8 \end{array} \right. \end{array} \quad \begin{array}{l} (1) \\ (2) \end{array}$$

$$\text{Clearing (1) of fractions,} \quad x - 2y = 2 \quad (3)$$

$$\text{Clearing (2) of fractions,} \quad 2x - y = 16 \quad (4)$$

$$\text{From (3),} \quad x = 2 + 2y \quad (5)$$

$$\text{From (4),} \quad x = \frac{16 + y}{2} \quad (6)$$

$$\text{By Ax. 7,} \quad 2 + 2y = \frac{16 + y}{2} \quad (7)$$

$$\text{Clearing of fractions,} \quad 4 + 4y = 16 + y \quad (8)$$

$$\text{Transposing and uniting,} \quad 3y = 12 \quad (9)$$

$$\text{Whence,} \quad y = 4 \quad (10)$$

$$\text{Substituting in (5),} \quad x = 2 + 8 = 10 \quad (11)$$

11. Given

$$\left\{ \begin{array}{l} \frac{x+y}{10} + \frac{x-y}{2} = 0 \\ \frac{x+y}{5} + \frac{x-y}{2} = 1 \end{array} \right. \quad (1)$$

$$\left\{ \begin{array}{l} \frac{x+y}{10} + \frac{x-y}{2} = 0 \\ \frac{x+y}{5} + \frac{x-y}{2} = 1 \end{array} \right. \quad (2)$$

Clear. (1) of fractions, $x + y + 5x - 5y = 0$ (3)

Clear. (2) of fractions, $2x + 2y + 5x - 5y = 10$ (4)

Reducing (3), $3x - 2y = 0$ (5)

Reducing (4), $7x - 3y = 10$ (6)

From (5), $x = \frac{2y}{3}$ (7)

From (6), $x = \frac{10 + 3y}{7}$ (8)

By Ax. 7, $\frac{2y}{3} = \frac{10 + 3y}{7}$ (9)

Clearing of fractions, $14y = 30 + 9y$ (10)

Transposing and uniting, $5y = 30$ (11)

Whence, $y = 6$ (12)

Substituting in (7), $x = \frac{2 \times 6}{3} = 4$ (13)

12. Given

$$\left\{ \begin{array}{l} \frac{x+8}{4} + 6y = 21 \\ \frac{y+6}{3} = 23 - 5x \end{array} \right. \quad (1)$$

$$\left\{ \begin{array}{l} \frac{x+8}{4} + 6y = 21 \\ \frac{y+6}{3} = 23 - 5x \end{array} \right. \quad (2)$$

Clearing (1) of fractions, $x + 8 + 24y = 84$ (3)

Clearing (2) of fractions, $y + 6 = 69 - 15x$ (4)

Reducing (3), $x + 24y = 76$ (5)

Reducing (4), $15x + y = 63$ (6)

From (5), $x = 76 - 24y$ (7)

From (6), $x = \frac{63 - y}{15}$ (8)

By Ax. 7, $76 - 24y = \frac{63 - y}{15}$ (9)

Clearing of fractions, $1140 - 360y = 63 - y$ (10)

Transposing and uniting, $-359y = -1077$ (11)

Whence, $y = 3$ (12)

Substituting in (8), $x = \frac{63 - 3}{15} = 4$ (13)

(ART. 173, pp. 140-142.)

$$\begin{array}{rcl} 3. \text{ Given} & \begin{cases} x + 4y = 16 \\ 4x + y = 34 \end{cases} & \begin{array}{l} (1) \\ (2) \end{array} \end{array}$$

$$\text{From (1),} \quad x = 16 - 4y \quad (3)$$

$$\text{Substituting in (2),} \quad 4(16 - 4y) + y = 34 \quad (4)$$

$$\text{Expanding,} \quad 64 - 16y + y = 34 \quad (5)$$

$$\text{Transposing and uniting,} \quad -15y = -30 \quad (6)$$

$$\text{Whence,} \quad y = 2 \quad (7)$$

$$\text{Substituting in (3),} \quad x = 16 - 8 = 8 \quad (8)$$

$$\begin{array}{rcl} 4. \text{ Given} & \begin{cases} x + 2y = 18 \\ 2x - y = 1 \end{cases} & \begin{array}{l} (1) \\ (2) \end{array} \end{array}$$

$$\text{From (1),} \quad x = 18 - 2y \quad (3)$$

$$\text{Substituting in (2),} \quad 2(18 - 2y) - y = 1 \quad (4)$$

$$\text{Expanding,} \quad 36 - 4y - y = 1 \quad (5)$$

$$\text{Transposing and uniting,} \quad -5y = -35 \quad (6)$$

$$\text{Whence,} \quad y = 7 \quad (7)$$

$$\text{Substituting in (3),} \quad x = 18 - 14 = 4 \quad (8)$$

$$\begin{array}{rcl} 5. \text{ Given} & \begin{cases} x + y = 13 \\ x - y = 3 \end{cases} & \begin{array}{l} (1) \\ (2) \end{array} \end{array}$$

$$\text{From (1),} \quad x = 13 - y \quad (3)$$

$$\text{Substituting in (2),} \quad 13 - y - y = 3 \quad (4)$$

$$\text{Transposing and uniting,} \quad -2y = -10 \quad (5)$$

$$\text{Whence,} \quad y = 5 \quad (6)$$

$$\text{Substituting in (3),} \quad x = 13 - 5 = 8 \quad (7)$$

$$\begin{array}{rcl} 6. \text{ Given} & \begin{cases} \frac{x}{2} - y = 1 \\ x - \frac{y}{2} = 8 \end{cases} & \begin{array}{l} (1) \\ (2) \end{array} \end{array}$$

$$\text{Clearing (1) of fractions,} \quad x - 2y = 2 \quad (3)$$

$$\text{From (3),} \quad x = 2 + 2y \quad (4)$$

$$\text{Substituting in (2),} \quad 2 + 2y - \frac{y}{2} = 8 \quad (5)$$

$$\text{Clearing (5) of fractions,} \quad 4 + 4y - y = 16 \quad (6)$$

$$\text{Transposing and uniting,} \quad 3y = 12 \quad (7)$$

$$\text{Whence,} \quad y = 4 \quad (8)$$

$$\text{Substituting in (4),} \quad x = 2 + 8 = 10 \quad (9)$$

$$\begin{array}{rcl} 7. \text{ Given} & \begin{cases} 3x + 5y = 40 \\ x + 2y = 14 \end{cases} & \begin{array}{l} (1) \\ (2) \end{array} \end{array}$$

$$\text{From (2),} \quad x = 14 - 2y \quad (3)$$

$$\text{Substituting in (1),} \quad 3(14 - 2y) + 5y = 40 \quad (4)$$

$$\text{Expanding,} \quad 42 - 6y + 5y = 40 \quad (5)$$

$$\text{Transposing and uniting,} \quad -y = -2 \quad (6)$$

$$\text{Whence,} \quad y = 2 \quad (7)$$

$$\text{Substituting in (3),} \quad x = 14 - 4 = 10 \quad \left. \vphantom{\begin{array}{l} (7) \\ (6) \end{array}} \right\} \text{Ans.} \quad (8)$$

$$\begin{array}{rcl} 8. \text{ Given} & \begin{cases} 5x + 3y = 0 \\ x - y = 8 \end{cases} & \begin{array}{l} (1) \\ (2) \end{array} \end{array}$$

$$\text{From (2),} \quad x = 8 + y \quad (3)$$

$$\text{Substituting in (1),} \quad 5(8 + y) + 3y = 0 \quad (4)$$

$$\text{Expanding,} \quad 40 + 5y + 3y = 0 \quad (5)$$

$$\text{Transposing and uniting,} \quad 8y = -40 \quad (6)$$

$$\text{Whence,} \quad y = -5 \quad (7)$$

$$\text{Substituting in (3),} \quad x = 8 + (-5) = 3 \quad (8)$$

$$\begin{array}{rcl} 9. \text{ Given} & \begin{cases} 6x + 5y = 77 \\ 4x - 3y = 7 \end{cases} & \begin{array}{l} (1) \\ (2) \end{array} \end{array}$$

$$\text{From (1),} \quad x = \frac{77 - 5y}{6} \quad (3)$$

$$\text{Substituting in (2),} \quad 4\left(\frac{77 - 5y}{6}\right) - 3y = 7 \quad (4)$$

$$\text{Expanding,} \quad \frac{154 - 10y}{3} - 3y = 7 \quad (5)$$

$$\text{Clearing of fractions,} \quad 154 - 10y - 9y = 21 \quad (6)$$

$$\text{Transposing and uniting,} \quad -19y = -133 \quad (7)$$

Whence, $y = 7$ (8)

Substituting in (3), $x = \frac{77 - 35}{6} = 7$ (9)

10. Given $\begin{cases} \frac{x}{3} + \frac{2y}{5} = 6 & (1) \\ \frac{2x}{3} + \frac{y}{5} = 6 & (2) \end{cases}$

From (1), $\frac{x}{3} = 6 - \frac{2y}{5}$ (3)

Substituting in (2), $2 \left(6 - \frac{2y}{5} \right) + \frac{y}{5} = 6$ (4)

Expanding, $12 - \frac{4y}{5} + \frac{y}{5} = 6$ (5)

Transposing and uniting, $-\frac{3y}{5} = -6$ (6)

Multiplying by $-\frac{5}{3}$, $y = 10$ (7)

Substituting in (3), $\frac{x}{3} = 6 - \frac{20}{5} = 2$ (8)

Multiplying by 3, $x = 6$ (9)

Perhaps the pupil could more readily solve these equations as follows :

Clearing (1) of fractions, $5x + 6y = 90$ (3)

Clearing (2) of fractions, $10x + 3y = 90$ (4)

From (3), $x = \frac{90 - 6y}{5}$ (5)

Substituting in (4), $180 - 12y + 3y = 90$ (6)

Transposing and uniting, $-9y = -90$ (7)

Whence, $y = 10$ (8)

Substituting in (5), $x = \frac{90 - 60}{5} = 6$ (9)

11. Given $\begin{cases} \frac{x+2}{3} + 8y = 31 & (1) \\ \frac{y+5}{4} + 10x = 192 & (2) \end{cases}$

Clearing (1) of fractions, $x + 2 + 24y = 93$ (3)

$$\text{Clearing (2) of fractions, } y + 5 + 40x = 768 \quad (4)$$

$$\text{Reducing (3), } x + 24y = 91 \quad (5)$$

$$\text{Reducing (4), } 40x + y = 763 \quad (6)$$

$$\text{From (5), } x = 91 - 24y \quad (7)$$

$$\text{Substituting in (6), } 40(91 - 24y) + y = 763 \quad (8)$$

$$\text{Expanding, } 3640 - 960y + y = 763 \quad (9)$$

$$\text{Transposing and uniting, } -959y = -2877 \quad (10)$$

$$\text{Whence, } y = 3 \quad (11)$$

$$\text{Substituting in (7), } x = 91 - 72 = 19 \quad (12)$$

$$12. \text{ Given } \begin{cases} \frac{x}{3} + \frac{y}{5} - 5 = 0 & (1) \\ 2x + \frac{y}{8} - 17 = 0 & (2) \end{cases}$$

$$\text{Clear. (1) of fract. and transp., } 5x + 3y = 75 \quad (3)$$

$$\text{Clear. (2) of fract. and transp., } 6x + y = 51 \quad (4)$$

$$\text{From (4), } y = 51 - 6x \quad (5)$$

$$\text{Substituting in (3), } 5x + 3(51 - 6x) = 75 \quad (6)$$

$$\text{Expanding, } 5x + 153 - 18x = 75 \quad (7)$$

$$\text{Transposing and uniting, } -13x = -78 \quad (8)$$

$$\text{Whence, } x = 6 \quad (9)$$

$$\text{Substituting in (5), } y = 51 - 36 = 15 \quad (10)$$

$$13. \text{ Given } \begin{cases} \frac{x+y}{8} - \frac{x-y}{2} = 0 & (1) \\ \frac{x+y}{4} + \frac{x-2y}{4} = 1\frac{1}{2} & (2) \end{cases}$$

$$\text{Clear. (1) of fract., } x + y - 4x + 4y = 0 \quad (3)$$

$$\text{Clear. (2) of fract., } x + y + x - 2y = 7 \quad (4)$$

$$\text{Uniting terms in (3), } -3x + 5y = 0 \quad (5)$$

$$\text{Uniting terms in (4), } 2x - y = 7 \quad (6)$$

$$\text{From (6), } y = 2x - 7 \quad (7)$$

$$\text{Substituting in (5), } -3x + 5(2x - 7) = 0 \quad (8)$$

$$\text{Expanding, } -3x + 10x - 35 = 0 \quad (9)$$

$$\text{Transposing and uniting, } 7x = 35 \quad (10)$$

$$\text{Whence, } x = 5 \quad (11)$$

$$\text{Substituting in (7), } y = 10 - 7 = 3 \quad (12)$$

(ART. 174, pp. 142-144.)

$$\begin{array}{rcl}
 3. \text{ Given} & \begin{cases} 4x + 3y = 25 \\ 12x - 6y = 30 \end{cases} & \begin{matrix} (1) \\ (2) \end{matrix} \\
 \text{Equation (1),} & 4x + 3y = 25 & \\
 \text{Dividing (2) by 2,} & 6x - 3y = 15 & (3) \\
 \text{Adding (1) and (3),} & 10x = 40 & (4) \\
 \text{Whence,} & x = 4 & (5) \\
 \text{Substituting in (1),} & 16 + 3y = 25 & (6) \\
 \text{Transposing and uniting,} & 3y = 9 & (7) \\
 \text{Whence,} & y = 3 & (8)
 \end{array}$$

$$\begin{array}{rcl}
 4. \text{ Given} & \begin{cases} 3x - y = 22 \\ 2x + 4y = 24 \end{cases} & \begin{matrix} (1) \\ (2) \end{matrix} \\
 \text{Multiplying (1) by 2,} & 6x - 2y = 44 & (3) \\
 \text{Dividing (2) by 2,} & x + 2y = 12 & (4) \\
 \text{Adding (3) and (4),} & 7x = 56 & (5) \\
 \text{Whence,} & x = 8 & (6) \\
 \text{Substituting in (1),} & 24 - y = 22 & (7) \\
 \text{Transposing and uniting,} & -y = -2 & (8) \\
 \text{Or,} & y = 2 & (9)
 \end{array}$$

$$\begin{array}{rcl}
 5. \text{ Given} & \begin{cases} x + 8y = 44 \\ 6x + y = 29 \end{cases} & \begin{matrix} (1) \\ (2) \end{matrix} \\
 \text{Multiplying (1) by 6,} & 6x + 48y = 264 & (3) \\
 \text{Subtracting (2) from (3),} & 47y = 235 & (4) \\
 \text{Whence,} & y = 5 & (5) \\
 \text{Substituting in (1),} & x + 40 = 44 & (6) \\
 \text{Transposing and uniting,} & x = 4 & (7)
 \end{array}$$

$$\begin{array}{rcl}
 6. \text{ Given} & \begin{cases} 23x - 8y = 70 \\ 8x - 2y = 40 \end{cases} & \begin{matrix} (1) \\ (2) \end{matrix} \\
 \text{Multiplying (2) by 4,} & 32x - 8y = 160 & (3) \\
 \text{Equation (1),} & 23x - 8y = 70 & \\
 \text{Subtracting (1) from (3),} & 9x = 90 & (4)
 \end{array}$$

Whence,	$x = 10$	(5)
Substituting in (2),	$80 - 2y = 40$	(6)
Transposing and uniting,	$-2y = -40$	(7)
Whence,	$y = 20$	(8)

7. Given	$\begin{cases} 4x - 5y = 0 \\ x - y = 1 \end{cases}$	(1)
	$\underline{x - y = 1}$	(2)
Multiplying (2) by 4,	$4x - 4y = 4$	(3)
Equation (1),	$4x - 5y = 0$	
	$\underline{4x - 5y = 0}$	
Subtracting (1) from (3),	$y = 4$ Ans.	(4)
Substituting in (2),	$x - 4 = 1$	(5)
Transposing and uniting,	$x = 5$ Ans.	(6)

8. Given	$\begin{cases} x + y = 35 \\ \frac{3x}{7} + \frac{9y}{14} = 18 \end{cases}$	(1)
	$\underline{\frac{3x}{7} + \frac{9y}{14} = 18}$	(2)
Dividing (2) by 3,	$\frac{x}{7} + \frac{3y}{14} = 6$	(3)
Clearing (3) of fractions,	$2x + 3y = 84$	(4)
Multiplying (1) by 2,	$2x + 2y = 70$	(5)
	$\underline{2x + 2y = 70}$	
Subtracting (5) from (4),	$y = 14$	(6)
Substituting in (1),	$x + 14 = 35$	(7)
Transposing and uniting,	$x = 21$	(8)

9. Given	$\begin{cases} \frac{7x}{8} + \frac{4y}{9} = 29 \\ \frac{11x}{12} - \frac{5y}{6} = 7 \end{cases}$	(1)
	$\underline{\frac{11x}{12} - \frac{5y}{6} = 7}$	(2)
Clearing (1) of fractions,	$63x + 32y = 2088$	(3)
Clearing (2) of fractions,	$11x - 10y = 84$	(4)
Multiplying (3) by 5,	$315x + 160y = 10440$	(5)
Multiplying (4) by 16,	$176x - 160y = 1344$	(6)
	$\underline{176x - 160y = 1344}$	
Adding (5) and (6),	$491x = 11784$	(7)
Whence,	$x = 24$	(8)

$$\text{Substituting in (4),} \quad 264 - 10y = 84 \quad (9)$$

$$\text{Transposing and uniting,} \quad -10y = -180 \quad (10)$$

$$\text{Whence,} \quad y = 18 \quad (11)$$

(ART. 175, pp. 144-146.)

$$\begin{array}{l} 1. \text{ Given} \\ \left\{ \begin{array}{l} 2x + y = 35 \\ 5x - 3y = 27 \end{array} \right. \end{array} \quad \begin{array}{l} (1) \\ (2) \end{array}$$

$$\text{From (1),} \quad y = 35 - 2x \quad (3)$$

$$\text{Substituting in (2), } 5x - 3(35 - 2x) = 27 \quad (4)$$

$$\text{Expanding,} \quad 5x - 105 + 6x = 27 \quad (5)$$

$$\text{Transposing and uniting,} \quad 11x = 132 \quad (6)$$

$$\text{Whence,} \quad x = 12 \quad (7)$$

$$\text{Substituting in (3),} \quad y = 35 - 24 = 11 \quad (8)$$

$$\begin{array}{l} 2. \text{ Given} \\ \left\{ \begin{array}{l} 5y - 5x = 15 \\ 3x + 5y = 71 \end{array} \right. \end{array} \quad \begin{array}{l} (1) \\ (2) \end{array}$$

$$\text{Subtracting (1) from (2),} \quad 8x = 56 \quad (3)$$

$$\text{Whence,} \quad x = 7 \quad (4)$$

$$\text{Dividing (1) by 5,} \quad y - x = 3 \quad (5)$$

$$\text{Substituting (4) in (5),} \quad y - 7 = 3 \quad (6)$$

$$\text{Whence,} \quad y = 10 \quad (7)$$

$$\begin{array}{l} 3. \text{ Given} \\ \left\{ \begin{array}{l} \frac{x}{6} + \frac{y}{4} = 6 \\ \frac{x}{4} + \frac{y}{6} = 5\frac{2}{3} \end{array} \right. \end{array} \quad \begin{array}{l} (1) \\ (2) \end{array}$$

$$\text{Clearing (1) of fractions,} \quad 2x + 3y = 72 \quad (3)$$

$$\text{Clearing (2) of fractions,} \quad 3x + 2y = 68 \quad (4)$$

$$\text{From (3),} \quad y = \frac{72 - 2x}{3} \quad (5)$$

$$\text{From (4),} \quad y = \frac{68 - 3x}{2} \quad (6)$$

$$\text{By Ax. 7,} \quad \frac{72 - 2x}{3} = \frac{68 - 3x}{2} \quad (7)$$

$$\text{Clearing of fractions,} \quad 144 - 4x = 204 - 9x \quad (8)$$

$$\text{Transposing and uniting,} \quad 5x = 60 \quad (9)$$

$$\text{Whence,} \quad x = 12 \quad (10)$$

$$\text{Substituting in (5),} \quad y = \frac{72 - 24}{3} = 16 \quad (11)$$

$$4. \text{ Given} \quad \begin{cases} 3x + 2y = 4 & (1) \\ 4y - 3x + 1 = 0 & (2) \end{cases}$$

$$\text{Adding (1) and (2),} \quad 6y + 1 = 4 \quad (3)$$

$$\text{Transposing and uniting,} \quad 6y = 3 \quad (4)$$

$$\text{Whence,} \quad y = \frac{1}{2} \quad (5)$$

$$\text{Substituting in (1),} \quad 3x + 1 = 4 \quad (6)$$

$$\text{Transposing and uniting,} \quad 3x = 3 \quad (7)$$

$$\text{Whence,} \quad x = 1 \quad (8)$$

$$5. \text{ Given} \quad \begin{cases} -4x + 3y = 45 & (1) \\ 2y + 6x = 4 & (2) \end{cases}$$

$$\text{Multiplying (1) by 3,} \quad -12x + 9y = 135 \quad (3)$$

$$\text{Multiplying (2) by 2,} \quad 12x + 4y = 8 \quad (4)$$

$$\text{Adding (3) and (4),} \quad 13y = 143 \quad (5)$$

$$\text{Whence,} \quad y = 11 \quad (6)$$

$$\text{Substituting in (2),} \quad 22 + 6x = 4 \quad (7)$$

$$\text{Transposing and uniting,} \quad 6x = -18 \quad (8)$$

$$\text{Whence,} \quad x = -3 \quad (9)$$

$$6. \text{ Given} \quad \begin{cases} \frac{x}{2} + y = 35 & (1) \\ \frac{y}{3} + x = 45 & (2) \end{cases}$$

$$\text{Clearing (1) of fractions,} \quad x + 2y = 70 \quad (3)$$

$$\text{Clearing (2) of fractions,} \quad 3x + y = 135 \quad (4)$$

$$\text{From (3),} \quad x = 70 - 2y \quad (5)$$

$$\text{Substituting in (4),} \quad 3(70 - 2y) + y = 135 \quad (6)$$

$$\text{Expanding,} \quad 210 - 6y + y = 135 \quad (7)$$

$$\text{Transposing and uniting,} \quad -5y = -75 \quad (8)$$

$$\begin{array}{ll} \text{Whence,} & y = 15 \quad (9) \\ \text{Substituting in (5),} & x = 70 - 30 = 40 \quad (10) \end{array}$$

$$\begin{array}{ll} 7. \text{ Given} & \begin{cases} 10x = 2 + 2y & (1) \\ 4y = 20 - 4x & (2) \end{cases} \end{array}$$

$$\text{Reducing (1),} \quad 5x - y = 1 \quad (3)$$

$$\text{Reducing (2),} \quad x + y = 5 \quad (4)$$

$$\text{Adding (3) and (4),} \quad 6x = 6 \quad (5)$$

$$\text{Whence,} \quad x = 1 \quad (6)$$

$$\text{Substituting in (4),} \quad 1 + y = 5 \quad (7)$$

$$\text{Whence,} \quad y = 4 \quad (8)$$

$$\begin{array}{ll} 8. \text{ Given} & \begin{cases} x + y = a & (1) \\ x - y = b & (2) \end{cases} \end{array}$$

$$\text{Adding (1) and (2),} \quad 2x = a + b \quad (3)$$

$$\text{Whence,} \quad x = \frac{a + b}{2} \quad (4)$$

$$\text{Subtracting (2) from (1),} \quad 2y = a - b \quad (5)$$

$$\text{Whence,} \quad y = \frac{a - b}{2} \quad (6)$$

$$\begin{array}{ll} 9. \text{ Given} & \begin{cases} 7x - 3y = 26 & (1) \\ 2x - 2y = 6\frac{2}{3} & (2) \end{cases} \end{array}$$

$$\text{Multiplying (1) by 2,} \quad 14x - 6y = 52 \quad (3)$$

$$\text{Multiplying (2) by 3,} \quad 6x - 6y = 20 \quad (4)$$

$$\text{Subtracting (4) from (3)} \quad 8x = 32 \quad (5)$$

$$\text{Whence,} \quad x = 4 \quad (6)$$

$$\text{Substituting in (4),} \quad 24 - 6y = 20 \quad (7)$$

$$\text{Transposing and uniting,} \quad -6y = -4 \quad (8)$$

$$\text{Whence,} \quad y = \frac{2}{3} \quad (9)$$

$$\begin{array}{ll} 10. \text{ Given} & \begin{cases} \frac{3x}{4} - \frac{y}{2} = 9 & (1) \\ 2x - 2y = 16 & (2) \end{cases} \end{array}$$

$$\text{Clearing (1) of fractions,} \quad 3x - 2y = 36 \quad (3)$$

$$\text{Subtracting (2) from (3),} \quad x = 20 \quad (4)$$

$$\text{Dividing (2) by 2,} \quad x - y = 8 \quad (5)$$

$$\text{Subtracting (5) from (4),} \quad y = 12 \quad (6)$$

11. Given

$$\begin{cases} \frac{x}{4} + \frac{y}{5} = 5 & (1) \\ \frac{2x}{3} + y = 18 & (2) \end{cases}$$

$$\text{Clearing (1) of fractions,} \quad 5x + 4y = 100 \quad (3)$$

$$\text{Clearing (2) of fractions,} \quad 2x + 3y = 54 \quad (4)$$

$$\text{From (3),} \quad x = \frac{100 - 4y}{5} \quad (5)$$

$$\text{From (4),} \quad x = \frac{54 - 3y}{2} \quad (6)$$

$$\text{By Ax. 7,} \quad \frac{100 - 4y}{5} = \frac{54 - 3y}{2} \quad (7)$$

$$\text{Clearing of fractions,} \quad 200 - 8y = 270 - 15y \quad (8)$$

$$\text{Transposing and uniting,} \quad 7y = 70 \quad (9)$$

$$\text{Whence,} \quad y = 10 \quad (10)$$

$$\text{Substituting in (6),} \quad x = \frac{54 - 30}{2} = 12 \quad (11)$$

NOTE. The last equations are readily solved by multiplying (2) by 4 and subtracting the resulting equation from (3), thus obtaining $\frac{7x}{8} = 28$.

12. Given

$$\begin{cases} 2y + 79 = 5x & (1) \\ 3x - 7 = 4 + x + y & (2) \end{cases}$$

$$\text{Transp. and uniting in (2),} \quad 2x - y = 11 \quad (3)$$

$$\text{From (3),} \quad y = 2x - 11 \quad (4)$$

$$\text{Substituting in (1),} \quad 4x - 22 + 79 = 5x \quad (5)$$

$$\text{Transposing and uniting,} \quad -x = -57 \quad (6)$$

$$\text{Whence,} \quad x = 57 \quad (7)$$

$$\text{Substituting in (4),} \quad y = 114 - 11 = 103 \quad (8)$$

13. Given

$$\begin{cases} \frac{1}{x} + \frac{2}{y} = \frac{11}{15} \end{cases} \quad (1)$$

$$\begin{cases} \frac{3}{x} + \frac{4}{y} = \frac{9}{5} \end{cases} \quad (2)$$

Multiplying (1) by 3,

$$\frac{3}{x} + \frac{6}{y} = \frac{11}{5} \quad (3)$$

Subtracting (2) from (3),

$$\frac{2}{y} = \frac{2}{5} \quad (4)$$

Dividing by 2,

$$\frac{1}{y} = \frac{1}{5} \quad (5)$$

Whence,

$$y = 5 \quad (6)$$

Substituting (4) in (1),

$$\frac{1}{x} + \frac{2}{5} = \frac{11}{15} \quad (7)$$

Transposing and uniting,

$$\frac{1}{x} = \frac{5}{15} = \frac{1}{3} \quad (8)$$

Whence,

$$x = 3 \quad (9)$$

14. Given

$$\begin{cases} 2x^{-1} + 3y^{-1} = \frac{3}{2} \end{cases} \quad (1)$$

$$\begin{cases} 4x^{-1} + 8y^{-1} = \frac{13}{2} \end{cases} \quad (2)$$

Multiplying (1) by 2,

$$4x^{-1} + 6y^{-1} = 3 \quad (3)$$

Subtracting (3) from (2),

$$2y^{-1} = \frac{1}{2} \quad (4)$$

Dividing by 2,

$$y^{-1} = \frac{1}{4} \quad (5)$$

Multiplying by 6 y,

$$6 = y \quad (6)$$

Or,

$$y = 6 \quad (7)$$

Substituting (5) in (1),

$$2x^{-1} + \frac{1}{2} = \frac{3}{2} \quad (8)$$

Transposing and uniting,

$$2x^{-1} = 1 \quad (9)$$

Dividing by 2,

$$x^{-1} = \frac{1}{2} \quad (10)$$

Whence,

$$x = 2 \quad (11)$$

NOTE. We might have divided (2) by 2, instead of multiplying (1) by 2.

It will be observed that examples 13 and 14 are similar to each other, in one case the fractional form being employed, and in the other negative exponents. The pupil, of course, already understands that x^{-1} and y^{-1} are the same as $\frac{1}{x}$ and $\frac{1}{y}$.

15. Given

$$\begin{cases} \frac{x}{2} + \frac{y}{3} = a & (1) \end{cases}$$

$$\begin{cases} \frac{x}{3} + \frac{y}{4} = b & (2) \end{cases}$$

$$\text{Clearing (1) of fractions,} \quad 3x + 2y = 6a \quad (3)$$

$$\text{Clearing (2) of fractions,} \quad 4x + 3y = 12b \quad (4)$$

$$\text{Multiplying (3) by 4,} \quad 12x + 8y = 24a \quad (5)$$

$$\text{Multiplying (4) by 3,} \quad 12x + 9y = 36b \quad (6)$$

$$\text{Subtracting (5) from (6),} \quad y = 36b - 24a \quad (7)$$

$$\text{Multiplying (3) by 3,} \quad 9x + 6y = 18a \quad (8)$$

$$\text{Multiplying (4) by 2,} \quad 8x + 6y = 24b \quad (9)$$

$$\text{Subtracting (9) from (8),} \quad x = 18a - 24b \quad (10)$$

16. Given

$$\begin{cases} \frac{2x}{5} + \frac{3y}{4} = \frac{9}{20} & (1) \end{cases}$$

$$\begin{cases} \frac{3x}{4} + \frac{2y}{5} = \frac{61}{120} & (2) \end{cases}$$

$$\text{Clearing (1) of fractions,} \quad 8x + 15y = 9 \quad (3)$$

$$\text{Multiplying (2) by 20,} \quad 15x + 8y = \frac{61}{6} \quad (4)$$

$$\text{Multiplying (3) by 15,} \quad 120x + 225y = 135 \quad (5)$$

$$\text{Multiplying (4) by 8,} \quad 120x + 64y = \frac{248}{3} \quad (6)$$

$$\text{Subtracting (6) from (5),} \quad 161y = \frac{161}{3} \quad (7)$$

$$\text{Dividing by 161,} \quad y = \frac{1}{3} \quad (8)$$

$$\text{Substituting in (3),} \quad 8x + 5 = 9 \quad (9)$$

$$\text{Transposing and uniting,} \quad 8x = 4 \quad (10)$$

$$\text{Whence,} \quad x = \frac{1}{2} \quad (11)$$

NOTE. Equation (2) may be wholly cleared of fractions, if it is preferred, producing $90x + 48y = 61$ instead of (4). From this and equation (3), either x or y is readily eliminated.

17. Given

$$\begin{cases} \frac{4x+y}{10} + x - 5y = 33\frac{1}{2} & (1) \end{cases}$$

$$\begin{cases} \frac{8x+12y}{15} + 9y = 1 & (2) \end{cases}$$

$$\text{Clearing (1) of fractions,} \quad 8x + 2y + 20x - 100y = 665 \quad (3)$$

$$\text{Clearing (2) of fractions, } 8x + 12y + 135y = 15 \quad (4)$$

$$\text{Uniting terms in (3), } 28x - 98y = 665 \quad (5)$$

$$\text{Dividing (5) by 7, } 4x - 14y = 95 \quad (6)$$

$$\text{Transp. and uniting in (4), } 8x + 147y = 15 \quad (7)$$

$$\text{From (6), } x = \frac{95 + 14y}{4} \quad (8)$$

$$\text{From (7), } x = \frac{15 - 147y}{8} \quad (9)$$

$$\text{By Ax. 7, } \frac{95 + 14y}{4} = \frac{15 - 147y}{8} \quad (10)$$

$$\text{Clearing of fractions, } 190 + 28y = 15 - 147y \quad (11)$$

$$\text{Transposing and uniting, } 175y = -175 \quad (12)$$

$$\text{Whence, } y = -1 \quad (13)$$

$$\text{Substituting in (8), } x = \frac{95 - 14}{4} = 20\frac{1}{4} \quad (14)$$

NOTE. It is evident that x can be very readily eliminated from (6) and (7) by subtraction, after multiplying (6) by 2.

The student may find the third method of elimination so much easier for him as to incline him to use it altogether. He should, however, occasionally be required to employ the other methods of elimination, and be able to solve any example by either of the methods.

PROBLEMS

LEADING TO SIMPLE EQUATIONS CONTAINING TWO UNKNOWN QUANTITIES.

(ART. 176, pp. 146 - 152.)

2. Let x = the greater number,
and y = the less number.

$$\text{Then, } x + y = 133 \quad (1)$$

$$\text{and } x - y = 47 \quad (2)$$

$$\text{Adding (1) and (2), } 2x = 180 \quad (3)$$

$$\text{Whence, } x = 90 \quad (4)$$

Subtracting (2) from (1), $2y = 86$ (5)

Whence, $y = 43$ (6)

3. Let $x =$ the men's wages,
and $y =$ the boys' wages.

Then, $4x + 6y = 72$ (1)

and $3x + 9y = 81$ (2)

Dividing (1) by 2, $2x + 3y = 36$ (3)

Dividing (2) by 3, $x + 3y = 27$ (4)

Subtracting (4) from (3), $x = 9$ (5)

Substituting in (4), $9 + 3y = 27$ (6)

Transposing and uniting, $3y = 18$ (7)

Whence, $y = 6$ (8)

4. Let $x =$ value of first horse,
and $y =$ value of second horse.

Then, $x + 4y = 580$ (1)

and $4x + y = 520$ (2)

From (2), $y = 520 - 4x$ (3)

Substituting in (1), $x + 4(520 - 4x) = 580$ (4)

Expanding, $x + 2080 - 16x = 580$ (5)

Transposing and uniting, $-15x = -1500$ (6)

Whence, $x = 100$ (7)

Substituting in (3), $y = 520 - 400 = 120$ (8)

6. Let $x =$ figure in tens' place,
and $y =$ figure in units' place.
Then $10x + y =$ the number.

From the conditions, $10x + y = 4x + 4y$ (1)

and $20x + 2y - 9 = 10y + x$ (2)

Reducing (1), $2x = y$ (3)

Reducing (2), $19x - 8y = 9$ (4)

Substituting (3) in (4), $19x - 16x = 9$ (5)

Uniting terms, $3x = 9$ (6)

Whence, $x = 3$ (7)

Substituting in (3), $y = 3 \times 2 = 6$ (8)

Hence, $10x + y = 36$ (9)

7. Let
and

$x =$ lady's age,
 $y =$ gentleman's age.

Then, $x - 7 = 3(y - 7)$ (1)

and $x + 7 = 2(y + 7)$ (2)

Reducing (1), $x - 3y = -14$ (3)

Reducing (2), $x - 2y = 7$ (4)

Subtracting (3) from (4), $y = 21$ (5)

Substituting in (4), $x - 42 = 7$ (6)

Transposing and uniting, $x = 49$ (7)

8. Let
and

$x =$ A's money,
 $y =$ B's money.

Then, $\frac{1}{2}x + \frac{1}{4}y = 6$ (1)

and $\frac{1}{4}x + \frac{1}{8}y = 5\frac{1}{2}$ (2)

Clearing (1) of fractions, $2x + 3y = 72$ (3)

Clearing (2) of fractions, $3x + 2y = 68$ (4)

From (3), $y = \frac{72 - 2x}{3}$ (5)

From (4), $y = \frac{68 - 3x}{2}$ (6)

By Ax. 7, $\frac{72 - 2x}{3} = \frac{68 - 3x}{2}$ (7)

Clearing of fractions, $144 - 4x = 204 - 9x$ (8)

Transposing and uniting, $5x = 60$ (9)

Whence, $x = 12$ (10)

Substituting in (6), $y = \frac{68 - 36}{2} = 16$ (11)

9. Let $x =$ sum in one purse,
and $y =$ sum in other purse.

Then, $x + y = 84$ (1)

and $\frac{x}{y} = 13$ (2)

Clearing (2) of fractions, $x = 13y$ (3)

Substituting in (1), $13y + y = 84$ (4)

Uniting terms, $14y = 84$ (5)

Whence, $y = 6$ (6)

Substituting in (3), $x = 13 \times 6 = 78$ (7)

11. Let $x =$ James's age,
and $y =$ John's age.

Then, $x : y :: 3 : 4$ (1)

and $x + 6 : y + 6 :: 5 : 6$ (2)

From (1), $4x = 3y$ (3)

From (2), $6x + 36 = 5y + 30$ (4)

Transposing and uniting in (4), $6x - 5y = -6$ (5)

Multiplying (3) by 3, $12x - 9y = 0$ (6)

Multiplying (5) by 2, $12x - 10y = -12$ (7)

Subtracting (7) from (6), $y = 12$ (8)

Substituting in (3), $4x = 36$ (9)

Whence, $x = 9$ (10)

12. Let $x =$ greater number,
and $y =$ less number.

Then, $x : 24 :: x + y : 42$ (1)

and $x - y : 6 :: 4 : 3$ (2)

From (1), $42x = 24x + 24y$ (3)

From (2), $3x - 3y = 24$ (4)

Reducing (3), $3x - 4y = 0$ (5)

Subtracting (5) from (4), $y = 24$ (6)

$$\begin{array}{lll} \text{Substituting in (5),} & 3x = 96 & (7) \\ \text{Whence,} & x = 32 & (8) \end{array}$$

$$\begin{array}{lll} 14. \text{ Let} & x = \text{greater part,} \\ \text{and} & y = \text{less part.} \\ \text{Then,} & x + y = 72 & (1) \\ \text{and} & 3x - 2y = 121 & (2) \\ \text{Multiplying (1) by 2,} & 2x + 2y = 144 & (3) \\ \text{Adding (2) and (3),} & 5x = 265 & (4) \\ \text{Whence,} & x = 53 & (5) \\ \text{Substituting in (1),} & 53 + y = 72 & (6) \\ \text{Whence,} & y = 19 & (7) \end{array}$$

$$\begin{array}{lll} 15. \text{ Let} & x = \text{no. men at \$0.90,} \\ \text{and} & y = \text{no. men at \$1.50.} \\ \text{Then,} & x + y = 50 & (1) \\ \text{and} & 90x + 150y = 4800 & (2) \\ \text{Multiplying (1) by 3,} & 3x + 3y = 150 & (3) \\ \text{Dividing (2) by 30,} & 3x + 5y = 160 & (4) \\ \text{Subtracting (3) from (4),} & 2y = 10 & (5) \\ \text{Whence,} & y = 5 & (6) \\ \text{Substituting in (1),} & x + 5 = 50 & (7) \\ \text{Whence,} & x = 45 & (8) \end{array}$$

$$\begin{array}{lll} 16. \text{ Let} & x = \text{wages of each man,} \\ \text{and} & y = \text{wages of each woman.} \\ \text{Then,} & 5x + 7y = 1640 & (1) \\ \text{and} & 7x - 6y = 400 & (2) \\ \text{Multiplying (1) by 7,} & 35x + 49y = 11480 & (3) \\ \text{Multiplying (2) by 5,} & 35x - 30y = 2000 & (4) \\ \text{Subtracting (4) from (3),} & 79y = 9480 & (5) \end{array}$$

Whence, $y = 120$ (6)

Substituting in (1), $5x + 840 = 1640$ (7)

Transposing and uniting, $5x = 800$ (8)

Whence, $x = 160$ (9)

NOTE. In the last two solutions, the sums of money are expressed in cents throughout.

17. Let $x =$ the numerator,
and $y =$ the denominator.
Therefore $\frac{x}{y} =$ the fraction.

Then,
$$\frac{x + 4}{y} = \frac{1}{2} \quad (1)$$

and
$$\frac{x}{y + 7} = \frac{1}{5} \quad (2)$$

From (1), $2x - y = -8$ (3)

From (2), $5x - y = 7$ (4)

Subtracting (3) from (4), $3x = 15$ (5)

Whence, $x = 5$ (6)

Substituting in (4), $25 - y = 7$ (7)

Whence, $-y = -18$ (8)

Or, $y = 18$ (9)

Therefore,
$$\frac{x}{y} = \frac{5}{18} \quad (10)$$

19. Let $x =$ income tax,
and $y =$ assessed tax.

Then, $x + y = 30$ (1)

and
$$x + \frac{20x}{100} + y - \frac{25y}{100} = 32\frac{1}{2} \quad (2)$$

From (2),
$$\frac{24x}{20} + \frac{15y}{20} = \frac{257}{8} \quad (3)$$

Clearing (3) of fractions, $48x + 15y = 1285$ (4)

Multiplying (1) by 30, $30x + 30y = 900$ (5)

Subtracting (5) from (4), $18x = 385$ (6)

Whence, $x = 21\frac{7}{8}$ (7)

Substituting in (1), $21\frac{7}{8} + y = 30$ (8)

Whence, $y = 8\frac{1}{8}$ (9)

20. Let $x =$ the first quantity,
and $y =$ the second quantity.

Then, $x + a = m y$ (1)

and $y + b = n x$ (2)

From (1), $-x + m y = a$ (3)

Multiplying (2) by m , $m n x - m y = b m$ (4)

Adding (3) and (4), $m n x - x = a + b m$ (5)

Whence, $x = \frac{a + b m}{m n - 1}$ (6)

Multiplying (3) by n , $-n x + m n y = a n$ (7)

From (2), $n x - y = b$ (8)

Adding (7) and (8), $m n y - y = b + a n$ (9)

Whence, $y = \frac{b + a n}{m n - 1}$ (10)

21. Let $x =$ A's money,
and $y =$ B's money.

Then, $x = \frac{4}{5} y$ (1)

and $x + 10 = y - 10$ (2)

Substituting (1) in (2), $\frac{4}{5} y + 10 = y - 10$ (3)

Clearing (3) of fractions, $4 y + 90 = 5 y - 90$ (4)

Transposing and uniting, $-5 y = -180$ (5)

Whence, $y = 36$ (6)

Substituting in (1), $x = \frac{4}{5} \times 36 = 28\frac{4}{5}$ (7)

22. Let $x =$ time required by the man,
and $y =$ time required by his wife.

Then $\frac{1}{x} = x^{-1} =$ part consumed by man in 1 day,

$\frac{1}{y} = y^{-1} =$ part consumed by wife in 1 day,

and $\frac{1}{15} =$ part consumed by both in 1 day.

Hence, $x^{-1} + y^{-1} = \frac{1}{15}$ (1)

and $6x^{-1} + 6y^{-1} + 30y^{-1} = 1$ (2)

Multiplying (1) by 30, $30x^{-1} + 30y^{-1} = 2$ (3)

Multiplying (2) by 5, $30x^{-1} + 180y^{-1} = 5$ (4)

Subtracting (3) from (4), $150y^{-1} = 3$ (5)

Whence, $y^{-1} = \frac{1}{50}$ (6)

Therefore, $y = 50$ (7)

Substituting (6) in (1), $x^{-1} + \frac{1}{50} = \frac{1}{15}$ (8)

Whence, $x^{-1} = \frac{1}{15} - \frac{1}{50} = \frac{7}{150}$ (9)

Therefore, $x = \frac{150}{7} = 21\frac{3}{7}$ (10)

23. Let $x =$ no. lbs. at a cents,
 and $y =$ no. lbs. at b cents.
 Then $ax =$ cost of one kind,
 $by =$ cost of the other kind,
 and $cd =$ cost of the whole.

From the conditions, $x + y = d$ (1)

and $ax + by = cd$ (2)

Multiplying (1) by a , $ax + ay = ad$ (3)

Subtracting (2) from (3), $ay - by = ad - cd$ (4)

Whence, $y = \frac{d(a-c)}{a-b}$ (5)

Multiplying (1) by b , $bx + by = bd$ (6)

Equation (2), $ax + by = cd$ (7)

Subtracting (6) from (7), $ax - bx = cd - bd$ (8)

Whence, $x = \frac{d(c-b)}{a-b}$ (9)

24. Let $x =$ no. ounces copper,
 and $y =$ no. ounces tin.
 Then $\frac{x}{5\frac{1}{4}} = \frac{4x}{21} =$ no. cubic inches copper,
 and $\frac{y}{4\frac{1}{4}} = \frac{4y}{17} =$ no. cubic inches tin.

From the conditions, $x + y = 505$ ^{v3} (1)

and $\frac{4x}{21} + \frac{4y}{17} = 100$ (2)

From (2), $17x + 21y = 8925$ (3)

Multiplying (1) by 17, $17x + 17y = 8585$ (4)

Subtracting (4) from (3), $4y = 340$ (5)

Whence, $y = 85$ (6)

Substituting in (1), $x + 85 = 505$ (7)

Whence, $x = 420$ (8)

25. Let $x =$ the length,
 - and $y =$ the breadth.
 Then $xy =$ the contents.

From the conditions, $(x + 4)(y + 5) = xy + 116$ (1)

and $(x + 5)(y + 4) = xy + 113$ (2)

Expanding (1), $xy + 4y + 5x + 20 = xy + 116$ (3)

Expanding (2), $xy + 5y + 4x + 20 = xy + 113$ (4)

From (3), $5x + 4y = 96$ (5)

From (4), $4x + 5y = 93$ (6)

Multiplying (5) by 5, $25x + 20y = 480$ (7)

Multiplying (6) by 4, $16x + 20y = 372$ (8)

Subtracting (8) from (7), $9x = 108$ (9)

Whence, $x = 12$ (10)

Substituting in (5), $60 + 4y = 96$ (11)

Transposing and uniting, $4y = 36$ (12)

Whence, $y = 9$ (13)

26. Let $x =$ first sum,
 $x + 1000 =$ second sum,
 $x + 1500 =$ third sum ;
 and $y =$ first rate per cent,
 $y + 1 =$ second rate per cent,
 $y + 2 =$ third rate per cent.

Then

$$\frac{xy}{100} = \text{first income,}$$

$$\frac{(x + 1000)(y + 1)}{100} = \text{second income,}$$

$$\frac{(x + 1500)(y + 2)}{100} = \text{third income.}$$

$$\text{From the conditions, } \frac{(x + 1000)(y + 1)}{100} = \frac{xy}{100} + 80 \quad (1)$$

$$\text{and } \frac{(x + 1500)(y + 2)}{100} = \frac{xy}{100} + 150 \quad (2)$$

Expanding (1) and clearing of fractions,

$$xy + x + 1000y + 1000 = xy + 8000 \quad (3)$$

Expanding (2) and clearing of fractions,

$$xy + 2x + 1500y + 3000 = xy + 15000 \quad (4)$$

$$\text{From (3), } x + 1000y = 7000 \quad (5)$$

$$\text{From (4), } 2x + 1500y = 12000 \quad (6)$$

$$\text{Multiplying (5) by 2, } 2x + 2000y = 14000 \quad (7)$$

$$\text{Subtracting (6) from (7), } 500y = 2000 \quad (8)$$

$$\text{Whence, } y = 4 \quad (9)$$

$$\text{Therefore, } y + 1 = 5 \quad (10)$$

$$\text{Also, } y + 2 = 6 \quad (11)$$

$$\text{Substituting (9) in (5), } x + 4000 = 7000 \quad (12)$$

$$\text{Whence, } x = 3000 \quad (13)$$

$$\text{Therefore, } x + 1000 = 4000 \quad (14)$$

$$\text{Also, } x + 1500 = 4500 \quad (15)$$

SIMPLE EQUATIONS CONTAINING THREE OR MORE UNKNOWN QUANTITIES.

(ART. 177, pp. 153 - 156.)

3. Given

$$\begin{cases} x + 2y + z = 24 & (1) \\ 2x + y + 3z = 38 & (2) \\ 3x + 3y + 2z = 46 & (3) \end{cases}$$

$$\begin{array}{ll} \text{Multiplying (1) by 3,} & 3x + 6y + 3z = 72 \quad (4) \\ \text{Equation (2),} & 2x + y + 3z = 38 \end{array}$$

$$\text{Subtracting (2) from (4),} \quad x + 5y = 34 \quad (5)$$

$$\text{Multiplying (1) by 2,} \quad 2x + 4y + 2z = 48 \quad (6)$$

$$\text{Equation (3),} \quad 3x + 3y + 2z = 46$$

$$\text{Subtracting (3) from (6),} \quad -x + y = 2 \quad (7)$$

$$\text{Equation (5),} \quad x + 5y = 34$$

$$\text{Adding (7) and (5),} \quad 6y = 36 \quad (8)$$

$$\text{Whence,} \quad y = 6 \quad (9)$$

$$\text{Substituting in (5),} \quad x + 30 = 34 \quad (10)$$

$$\text{Whence,} \quad x = 4 \quad (11)$$

$$\text{Substituting (9) and (11) in (1),} \quad 4 + 12 + z = 24 \quad (12)$$

$$\text{Whence,} \quad z = 8 \quad (13)$$

$$\begin{array}{ll} 4. \text{ Given} & \left\{ \begin{array}{l} 4x + 2y - z = 26 \quad (1) \\ 5x + 2y - 3z = 16 \quad (2) \\ 2x - y + 2z = 23 \quad (3) \end{array} \right. \end{array}$$

FIRST SOLUTION.

$$\text{From (1),} \quad z = 4x + 2y - 26 \quad (4)$$

$$\text{Subst. in (2),} \quad 5x + 2y - 3(4x + 2y - 26) = 16 \quad (5)$$

$$\text{Subst. in (3),} \quad 2x - y + 2(4x + 2y - 26) = 23 \quad (6)$$

$$\text{Expanding (5),} \quad 5x + 2y - 12x - 6y + 78 = 16 \quad (7)$$

$$\text{Expanding (6),} \quad 2x - y + 8x + 4y - 52 = 23 \quad (8)$$

$$\text{From (7),} \quad 7x + 4y = 62 \quad (9)$$

$$\text{From (8),} \quad 10x + 3y = 75 \quad (10)$$

$$\text{From (9),} \quad y = \frac{62 - 7x}{4} \quad (11)$$

$$\text{Substituting in (10),} \quad 10x + 3\left(\frac{62 - 7x}{4}\right) = 75 \quad (12)$$

$$\text{Expand. and clear. fract.,} \quad 40x + 186 - 21x = 300 \quad (13)$$

$$\text{Transposing and uniting,} \quad 19x = 114 \quad (14)$$

$$\text{Whence,} \quad x = 6 \quad (15)$$

$$\text{Substituting in (11),} \quad y = \frac{62 - 42}{4} = 5 \quad (16)$$

$$\text{Substituting in (4),} \quad z = 24 + 10 - 26 = 8 \quad (17)$$

SECOND SOLUTION.

$$\text{Subtracting (2) from (1),} \quad -x + 2z = 10 \quad (4)$$

$$\text{Multiplying (3) by 2,} \quad 4x - 2y + 4z = 46 \quad (5)$$

$$\text{Adding (2) and (5),} \quad 9x + z = 62 \quad (6)$$

$$\text{Multiplying (6) by 2,} \quad 18x + 2z = 124 \quad (7)$$

$$\text{Subtracting (4) from (7),} \quad 19x = 114 \quad (8)$$

$$\text{Whence,} \quad x = 6 \quad (9)$$

$$\text{Substituting in (6),} \quad 54 + z = 62 \quad (10)$$

$$\text{Whence,} \quad z = 8 \quad (11)$$

$$\text{Substituting in (3),} \quad 12 - y + 16 = 23 \quad (12)$$

$$\text{Whence,} \quad y = 5 \quad (13)$$

NOTE. The first solution is given to illustrate elimination by substitution when there are three unknown quantities.

$$\begin{array}{l} 5. \text{ Given} \\ \left\{ \begin{array}{l} x + y + z = 33 \\ y - x + z = 23 \\ z - x - y = 1 \end{array} \right. \end{array} \quad \begin{array}{l} (1) \\ (2) \\ (3) \end{array}$$

$$\text{Subtracting (2) from (1),} \quad 2x = 10 \quad (4)$$

$$\text{Whence,} \quad x = 5 \quad (5)$$

$$\text{Subtracting (3) from (2),} \quad 2y = 22 \quad (6)$$

$$\text{Whence,} \quad y = 11 \quad (7)$$

$$\text{Adding (1) and (3),} \quad 2z = 34 \quad (8)$$

$$\text{Whence,} \quad z = 17 \quad (9)$$

NOTE. The above equations are peculiar, from the fact that two letters may be eliminated at once. This only happens when the coefficients of two letters may be made alike in two equations, and the signs either alike in the case of both letters, or different in the case of both.

$$\begin{array}{l} 7. \text{ Given} \\ \left\{ \begin{array}{l} x + y + z = 13 \\ u + x + y = 17 \\ u + x + z = 18 \\ u + y + z = 21 \end{array} \right. \end{array} \quad \begin{array}{l} (1) \\ (2) \\ (3) \\ (4) \end{array}$$

$$\text{Assume} \quad s = u + x + y + z \quad (5)$$

From (1),	$s - u = 13$	(6)
From (2),	$s - z = 17$	(7)
From (3),	$s - y = 18$	(8)
From (4),	$s - x = 21$	(9)
Adding, and substituting (5),	$4s - s = 69$	(10)
Or,	$3s = 69$	(11)
Whence,	$s = 23$	(12)
Substituting in (6),	$23 - u = 13$	(13)
Whence,	$u = 10$	(14)
Substituting in (7),	$23 - z = 17$	(15)
Whence,	$z = 6$	(16)
Substituting in (8),	$23 - y = 18$	(17)
Whence,	$y = 5$	(18)
Substituting in (9),	$23 - x = 21$	(19)
Whence,	$x = 2$	(20)

The above equations may also be solved as follows :

Adding (1), (2), (3), and (4),	$3u + 3x + 3y + 3z = 69$	(5)
Dividing by 3,	$u + x + y + z = 23$	(6)
Equation (1),	$x + y + z = 13$	
Subtracting (1) from (6),	$u = 10$	(7)
Subtracting (2) from (6),	$z = 6$	(8)
Subtracting (3) from (6),	$y = 5$	(9)
Subtracting (4) from (6),	$x = 2$	(10)

$$\begin{array}{lcl}
 \text{8. Given} & \left\{ \begin{array}{l} 7x - 3y - z = 12 \\ x + 2y + 3z = 17 \\ 4x - y + 2z = 13 \end{array} \right. & \begin{array}{l} (1) \\ (2) \\ (3) \end{array}
 \end{array}$$

$$\text{From (1),} \quad z = 7x - 3y - 12 \quad (4)$$

$$\text{From (2),} \quad z = \frac{17 - x - 2y}{3} \quad (5)$$

$$\text{From (3),} \quad z = \frac{13 - 4x + y}{2} \quad (6)$$

$$\text{Comparing (4) and (5), } 7x - 3y - 12 = \frac{17 - x - 2y}{3} \quad (7)$$

$$\text{Comparing (4) and (6), } 7x - 3y - 12 = \frac{13 - 4x + y}{2} \quad (8)$$

$$\text{Clear. (7) of fract., } 21x - 9y - 36 = 17 - x - 2y \quad (9)$$

$$\text{Clear. (8) of fract., } 14x - 6y - 24 = 13 - 4x + y \quad (10)$$

$$\text{Transp. and uniting in (9), } 22x - 7y = 53 \quad (11)$$

$$\text{Transp. and unit. in (10), } 18x - 7y = 37 \quad (12)$$

$$\text{Subtracting (12) from (11), } 4x = 16 \quad (13)$$

$$\text{Whence, } x = 4 \quad (14)$$

$$\text{Substituting in (12), } 72 - 7y = 37 \quad (15)$$

$$\text{Transposing and uniting, } -7y = -35 \quad (16)$$

$$\text{Whence, } y = 5 \quad (17)$$

$$\text{Substitut. in (4), } z = 28 - 15 - 12 = 1 \quad (18)$$

NOTE. In eliminating the last time, the method of comparison is abandoned, because there is so good an opportunity to eliminate y by subtraction.

$$\begin{array}{ll} 9. \text{ Given} & \left\{ \begin{array}{l} x + y - z = 0 \\ x + z - y = 2 \\ y + z - x = 4 \end{array} \right. \end{array} \quad \begin{array}{l} (1) \\ (2) \\ (3) \end{array}$$

$$\text{Adding (1), (2), and (3), } x + y + z = 6 \quad (4)$$

$$\text{Subtracting (1) from (4), } 2z = 6 \quad (5)$$

$$\text{Whence, } z = 3 \quad (6)$$

$$\text{Subtracting (2) from (4), } 2y = 4 \quad (7)$$

$$\text{Whence, } y = 2 \quad (8)$$

$$\text{Subtracting (3) from (4), } 2x = 2 \quad (9)$$

$$\text{Whence, } x = 1 \quad (10)$$

NOTE. A still simpler plan for elimination would be to add the original equations, taken two and two. Thus (1) + (2) gives the value of x , (1) + (3) gives y , and (2) + (3) gives z .

$$\begin{array}{ll} 10. \text{ Given} & \left\{ \begin{array}{l} \frac{x}{a} + \frac{y}{b} = 1 \\ \frac{x}{a} + \frac{z}{c} = 1 \\ \frac{y}{b} + \frac{z}{c} = 1 \end{array} \right. \end{array} \quad \begin{array}{l} (1) \\ (2) \\ (3) \end{array}$$

Subtracting (3) from (2), $\frac{x}{a} - \frac{y}{b} = 0$ (4)

Equation (1), $\frac{x}{a} + \frac{y}{b} = 1$

Subtracting (4) from (1), $\frac{2y}{b} = 1$ (5)

Whence, $y = \frac{b}{2}$ (6)

Adding (4) and (1), $\frac{2x}{a} = 1$ (7)

Whence, $x = \frac{a}{2}$ (8)

Substituting in (2), $\frac{1}{2} + \frac{z}{c} = 1$ (9)

Whence, $z = \frac{c}{2}$ (10)

11. Given $\begin{cases} x + y = a & (1) \\ x + z = b & (2) \\ y + z = c & (3) \end{cases}$

Subtracting (3) from (2), $x - y = b - c$ (4)

Adding (1) and (4), $2x = a + b - c$ (5)

Whence, $x = \frac{1}{2}(a + b - c)$ (6)

Subtracting (4) from (1), $2y = a - b + c$ (7)

Whence, $y = \frac{1}{2}(a - b + c)$ (8)

Substituting in (3), $\frac{a + c - b}{2} + z = c$ (9)

Clearing of fractions, etc. $2z = c - a + b$ (10)

Whence, $z = \frac{1}{2}(b + c - a)$ (11)

NOTE. Examples 10 and 11 may also be solved by subtracting each equation from half the sum of the three.

12. Given $\begin{cases} \frac{x}{4} + \frac{y}{3} + \frac{z}{2} = 9 & (1) \\ \frac{2x}{3} - \frac{y}{2} + \frac{3z}{4} = 11 & (2) \\ \frac{3x}{4} + \frac{2y}{3} - \frac{z}{2} = 9 & (3) \end{cases}$

- Adding (1) and (3), $x + y = 18$ (4)
- Clearing (2) of fractions, $8x - 6y + 9z = 132$ (5)
- Clearing (3) of fractions, $9x + 8y - 6z = 108$ (6)
- Multiplying (5) by 2, $16x - 12y + 18z = 264$ (7)
- Multiplying (6) by 3, $27x + 24y - 18z = 324$ (8)
-
- Adding (7) and (8), $43x + 12y = 588$ (9)
- Multiplying (4) by 12, $12x + 12y = 216$ (10)
-
- Subtracting (10) from (9), $31x = 372$ (11)
- Whence, $x = 12$ (12)
- Substituting in (4), $12 + y = 18$ (13)
- Whence, $y = 6$ (14)
- Substituting in (1), $3 + 2 + \frac{z}{2} = 9$ (15)
- Transposing and uniting, $\frac{z}{2} = 4$ (16)
- Whence, $z = 8$ (17)

PROBLEMS

LEADING TO SIMPLE EQUATIONS CONTAINING THREE
OR MORE UNKNOWN QUANTITIES.

(ART. 178, pp. 157-160.)

1. Let $x =$ price of an orange,
and $y =$ price of an apple,
 $z =$ price of a pear.
- Then, $3x + y + 2z = 14$ (1)
- $4x + 3y + z = 17$ (2)
- and $x + 4y + 3z = 13$ (3)
-
- Multiplying (2) by 2, $8x + 6y + 2z = 34$ (4)
- Subtracting (1) from (4), $5x + 5y = 20$ (5)
- Dividing (5) by 5, $x + y = 4$ (6)
- Multiplying (2) by 3, $12x + 9y + 3z = 51$ (7)
- Equation (3), $x + 4y + 3z = 13$

$$\text{Subtracting (3) from (7), } 11x + 5y = 38 \quad (8)$$

$$\text{Equation (5), } 5x + 5y = 20$$

$$\text{Subtracting (5) from (8), } 6x = 18 \quad (9)$$

$$\text{Whence, } x = 3 \quad (10)$$

$$\text{Substituting in (6), } 3 + y = 4 \quad (11)$$

$$\text{Whence, } y = 1 \quad (12)$$

$$\text{Substituting in (2), } 12 + 3 + z = 17 \quad (13)$$

$$\text{Whence, } z = 2 \quad (14)$$

2. Let

$x = \text{Mary's part,}$

$y = \text{Isabel's part,}$

$z = \text{Jane's part,}$

and

$u = \text{Ellen's part.}$

$$\text{Then, } x + y + z + u = 100 \quad (1)$$

$$2y + 3u = 160 \quad (2)$$

$$3x + 2z = 90 \quad (3)$$

$$\text{and } 2x + u = 60 \quad (4)$$

$$\text{Adding (2) and (3), } 3x + 2y + 2z + 3u = 250 \quad (5)$$

$$\text{Mult. (1) by 2, } 2x + 2y + 2z + 2u = 200 \quad (6)$$

$$\text{Subtracting (6) from (5), } x + u = 50 \quad (7)$$

$$\text{Equation (4), } 2x + u = 60$$

$$\text{Subtracting (7) from (4), } x = 10 \quad (8)$$

$$\text{Substituting in (7), } 10 + u = 50 \quad (9)$$

$$\text{Whence, } u = 40 \quad (10)$$

$$\text{Substituting in (2), } 2y + 120 = 160 \quad (11)$$

$$\text{Transposing and uniting, } 2y = 40 \quad (12)$$

$$\text{Whence, } y = 20 \quad (13)$$

$$\text{Substituting in (1), } 10 + 20 + z + 40 = 100 \quad (14)$$

$$\text{Whence, } z = 30 \quad (15)$$

3. Given

$$\left\{ \begin{array}{l} \frac{7x}{16} + \frac{12y}{16} + \frac{4z}{16} = 8 \end{array} \right. \quad (1)$$

$$\left\{ \begin{array}{l} \frac{3x}{16} + \frac{3y}{16} + \frac{7z}{16} = \frac{15}{4} \end{array} \right. \quad (2)$$

$$\left\{ \begin{array}{l} \frac{6x}{16} + \frac{y}{16} + \frac{5z}{16} = \frac{17}{4} \end{array} \right. \quad (3)$$

$$\text{Clearing (1) of fractions,} \quad 7x + 12y + 4z = 128 \quad (4)$$

$$\text{Clearing (2) of fractions,} \quad 3x + 3y + 7z = 60 \quad (5)$$

$$\text{Clearing (3) of fractions,} \quad 6x + y + 5z = 68 \quad (6)$$

$$\text{Multiplying (5) by 4,} \quad 12x + 12y + 28z = 240 \quad (7)$$

$$\text{Subtracting (4) from (7),} \quad 5x + 24z = 112 \quad (8)$$

$$\text{Multiplying (6) by 3,} \quad 18x + 3y + 15z = 204 \quad (9)$$

$$\text{Subtracting (5) from (9),} \quad 15x + 8z = 144 \quad (10)$$

$$\text{Multiplying (10) by 3,} \quad 45x + 24z = 432 \quad (11)$$

$$\text{Subtracting (8) from (11),} \quad 40x = 320 \quad (12)$$

$$\text{Whence,} \quad x = 8 \quad (13)$$

$$\text{Substituting in (10),} \quad 120 + 8z = 144 \quad (14)$$

$$\text{Transposing and uniting,} \quad 8z = 24 \quad (15)$$

$$\text{Whence,} \quad z = 3 \quad (16)$$

$$\text{Substituting in (6),} \quad 48 + y + 15 = 68 \quad (17)$$

$$\text{Whence,} \quad y = 5 \quad (18)$$

4. Let

x = price of the chaise,

y = price of the horse,

and

z = price of the harness.

Then,

$$x + y + z = 400 \quad (1)$$

$$x = 4z \quad (2)$$

and

$$z = \frac{y}{3} \quad (3)$$

$$\text{Clearing (3) of fractions,} \quad y = 3z \quad (4)$$

$$\text{Subs. (2) and (4) in (1),} \quad 4z + 3z + z = 400 \quad (5)$$

$$\text{Or,} \quad 8z = 400 \quad (6)$$

$$\text{Whence,} \quad z = 50 \quad (7)$$

$$\text{Substituting in (2),} \quad x = 4 \times 50 = 200 \quad (8)$$

$$\text{Substituting in (4),} \quad y = 3 \times 50 = 150 \quad (9)$$

NOTE. It will be seen that this problem can be readily solved by the use of only one unknown quantity.

5. Let $x =$ the first number,
 $y =$ the second number,
and $z =$ the third number.

Then, $x + y + z = 324$ (1)

$y - x = z - y$ (2)

and $x : z :: 5 : 7$ (3)

Transposing in (2), $2y - x - z = 0$ (4)

Adding (4) and (1), $3y = 324$ (5)

Whence, $y = 108$ (6)

Substituting in (1), $x + 108 + z = 324$ (7)

Whence, $x + z = 216$ (8)

Multiplying (8) by 5, $5x + 5z = 1080$ (9)

From (3), $7x - 5z = 0$ (10)

Adding (9) and (10), $12x = 1080$ (11)

Whence, $x = 90$ (12)

Substituting in (8), $90 + z = 216$ (13)

Whence, $z = 126$ (14)

6. Let $x =$ the husband's age,
 $y =$ the wife's age,
and $z =$ the son's age.

Then, $x + z = y + 12$ (1)

$y + z = x + 8$ (2)

and $x + y + z = 92$ (3)

Transposing in (1), $x - y + z = 12$ (4)

Subtracting (4) from (3), $2y = 80$ (5)

Whence, $y = 40$ (6)

Transposing in (2), $-x + y + z = 8$ (7)

Subtracting (7) from (3), $2x = 84$ (8)

Whence, $x = 42$ (9)

Adding (4) and (7), $2z = 20$ (10)

Whence, $z = 10$ (11)

7. Let	$x =$ no. bush. wheat,	
	$y =$ no. bush. barley,	
and	$z =$ no. bush. oats.	
Then,	$x + y + z = 146$	(1)
	$y - x = 15$	(2)
and	$x + y = z$	(3)
Substituting (3) in (1),	$2z = 146$	(4)
Whence,	$z = 73$	(5)
Therefore, from (3),	$x + y = 73$	(6)
Equation (2),	$-x + y = 15$	
Adding (6) and (2),	$2y = 88$	(7)
Whence,	$y = 44$	(8)
Subtracting (2) from (6),	$2x = 58$	(9)
Whence,	$x = 29$	(10)

8. Let x , y , and z represent the number of days in which A, B, and C, respectively, could perform the whole work alone. Then $\frac{1}{x}$, $\frac{1}{y}$, and $\frac{1}{z}$, or x^{-1} , y^{-1} , and z^{-1} , will represent the part which each can perform in one day. Also, A and B together can do $\frac{1}{10}$ of the work in one day, A and C $\frac{1}{15}$, and B and C $\frac{1}{12}$.

Hence,	$x^{-1} + y^{-1} = \frac{1}{10}$	(1)
	$x^{-1} + z^{-1} = \frac{1}{15}$	(2)
and	$y^{-1} + z^{-1} = \frac{1}{12}$	(3)
Subtracting (2) from (1),	$y^{-1} - z^{-1} = \frac{1}{10} - \frac{1}{15} = \frac{1}{30}$	(4)
Adding (3) and (4),	$2y^{-1} = \frac{1}{30} + \frac{1}{12} = \frac{4}{60}$	(5)
Dividing by 2,	$y^{-1} = \frac{1}{15}$	(6)
Whence,	$y = \frac{15}{1} = 15$	(7)
Subtracting (4) from (3),	$2z^{-1} = \frac{1}{12} - \frac{1}{30} = \frac{1}{20}$	(8)
Dividing by 2,	$z^{-1} = \frac{1}{40}$	(9)
Whence,	$z = \frac{40}{1} = 40$	(10)
Substituting (6) in (1),	$x^{-1} + \frac{1}{15} = \frac{1}{10}$	(11)

Transposing, $x^{-1} = \frac{1}{8} - \frac{41}{720} = \frac{49}{720}$ (12)

Whence, $x = \frac{720}{49} = 14\frac{24}{49}$ (13)

NOTE. Elimination may also be conveniently effected, and the value of each unknown quantity found, by subtracting each of the original equations from half the sum of the three equations.

9. Let $x =$ digit in hundreds' place,
 $y =$ digit in tens' place,
 and $z =$ digit in units' place.
 Then $100x + 10y + z =$ the number.

From the conditions, $x + y + z = 9$ (1)

$100x + 10y + z - 198 = 100z + 10y + x$ (2)

and $\frac{100x + 10y + z}{x} = 108$ (3)

Transposing and uniting in (2), $99x - 99z = 198$ (4)

Dividing (4) by 99, $x - z = 2$ (5)

Clearing (3) of fractions, $100x + 10y + z = 108x$ (6)

Or, $-8x + 10y + z = 0$ (7)

Multiplying (1) by 10, $10x + 10y + 10z = 90$ (8)

Subtracting (7) from (8), $18x + 9z = 90$ (9)

Dividing (9) by 9, $2x + z = 10$ (10)

Equation (5), $x - z = 2$

Adding (10) and (5), $3x = 12$ (11)

Whence, $x = 4$ (12)

Substituting in (10), $8 + z = 10$ (13)

Whence, $z = 2$ (14)

Substituting in (1), $4 + y + 2 = 9$ (15)

Whence, $y = 3$ (16)

Therefore, $100x + 10y + z = 432$ (17)

10. Let $x =$ value of first horse,
 $y =$ value of second horse,
 and $z =$ value of third horse.

Then,	$x + 220 = y + z$	(1)
	$y + 220 = 2(x + z)$	(2)
and	$z + 220 = 3(x + y)$	(3)
From (1),	$-x + y + z = 220$	(4)
From (2),	$2x - y + 2z = 220$	(5)
Adding (4) and (5),	$x + 3z = 440$	(6)
From (3),	$3x + 3y - z = 220$	(7)
Multiplying (4) by 3,	$-3x + 3y + 3z = 660$	(8)
Subtracting (7) from (8),	$-6x + 4z = 440$	(9)
Dividing (9) by 2,	$-3x + 2z = 220$	(10)
Multiplying (6) by (3),	$3x + 9z = 1320$	(11)
Adding (10) and (11),	$11z = 1540$	(12)
Whence,	$z = 140$	(13)
Substituting in (6),	$x + 420 = 440$	(14)
Whence,	$x = 20$	(15)
Substituting in (4),	$-20 + y + 140 = 220$	(16)
Whence,	$y = 100$	(17)

11. Let x , y , and z denote the number of days in which A, B, and C, respectively, can reap the field alone. Then, $\frac{1}{x}$, $\frac{1}{y}$, and $\frac{1}{z}$, or x^{-1} , y^{-1} , and z^{-1} will denote the part of the field that each can reap in one day. Also, A and B together can reap $\frac{1}{a}$ of the field in one day, A and C $\frac{1}{b}$, and B and C $\frac{1}{c}$.

$$\text{Hence, } x^{-1} + y^{-1} = a^{-1} \quad (1)$$

$$x^{-1} + z^{-1} = b^{-1} \quad (2)$$

$$\text{and } y^{-1} + z^{-1} = c^{-1} \quad (3)$$

Adding (1), (2), and (3),

$$2x^{-1} + 2y^{-1} + 2z^{-1} = a^{-1} + b^{-1} + c^{-1} \quad (4)$$

$$\text{Multiplying (3) by 2, } 2y^{-1} + 2z^{-1} = 2c^{-1} \quad (5)$$

Subtracting (5) from (4), $2x^{-1} = a^{-1} + b^{-1} - c^{-1}$ (6)

Multiplying by $a b c x$ (Art. 153, Note 3),

$$2abc = (bc + ac - ab)x \quad (7)$$

Whence,

$$x = \frac{2abc}{ac + bc - ab} \quad (8)$$

Equation (4), $2x^{-1} + 2y^{-1} + 2z^{-1} = a^{-1} + b^{-1} + c^{-1}$

Multiplying (2) by 2, $2x^{-1} + 2z^{-1} = 2b^{-1}$ (9)

Subtracting (9) from (4), $2y^{-1} = a^{-1} - b^{-1} + c^{-1}$ (10)

Multiplying by $abc y$, $2abc = (bc - ac + ab)y$ (11)

Whence,

$$y = \frac{2abc}{ab + bc - ac} \quad (12)$$

Equation (4), $2x^{-1} + 2y^{-1} + 2z^{-1} = a^{-1} + b^{-1} + c^{-1}$

Multiplying (1) by 2, $2x^{-1} + 2y^{-1} = 2a^{-1}$ (13)

Subtracting (13) from (4), $2z^{-1} = b^{-1} + c^{-1} - a^{-1}$ (14)

Multiplying by $abc z$, $2abc = (ac + ab - bc)z$ (15)

Whence,

$$z = \frac{2abc}{ab + ac - bc} \quad (16)$$

NOTE. Equations (6), (10), and (14) may be readily obtained by the method adopted in the solution of Ex. 11, Art. 177, or Prob. 8 of this Art.

12. Let

$x =$ the first number,

$y =$ the second number,

and

$z =$ the third number.

Then,

$$\frac{1}{2}x + \frac{1}{3}y + \frac{1}{4}z = 62 \quad (1)$$

$$\frac{1}{3}x + \frac{1}{4}y + \frac{1}{5}z = 47 \quad (2)$$

and

$$\frac{1}{4}x + \frac{1}{5}y + \frac{1}{6}z = 38 \quad (3)$$

Multiplying (1) by 120, $60x + 40y + 30z = 7440$ (4)

Multiplying (2) by 180, $60x + 45y + 36z = 8460$ (5)

Multiplying (3) by 240, $60x + 48y + 40z = 9120$ (6)

Subtracting (5) from (6), $3y + 4z = 660$ (7)

Subtracting (4) from (5), $5y + 6z = 1020$ (8)

Multiplying (7) by 5, $15y + 20z = 3300$ (9)

Multiplying (8) by 3, $15y + 18z = 3060$ (10)

Subtracting (10) from (9),	$2z = 240$	(11)
Whence,	$z = 120$	(12)
Substituting in (7),	$3y + 480 = 660$	(13)
Transposing and uniting,	$3y = 180$	(14)
Whence,	$y = 60$	(15)
Substituting in (1),	$\frac{1}{2}x + 20 + 30 = 62$	(16)
Transposing and uniting,	$\frac{1}{2}x = 12$	(17)
Whence,	$x = 24$	(18)

NOTE. Instead of first clearing of fractions and then rendering the coefficients of x the same, we have here used such multipliers as will effect both objects at once. The student may, however, find it easier to keep the two distinct.

13. Let	$x = A's \text{ money,}$	
	$y = B's \text{ money,}$	
and	$z = C's \text{ money.}$	
Then,	$x + \frac{2}{3}y = 219$	(1)
	$y + \frac{1}{2}z = 219$	(2)
and	$z + \frac{2}{3}x = 219$	(3)
Clearing (1) of fractions,	$7x + 3y = 1533$	(4)
Clearing (2) of fractions,	$9y + 5z = 1971$	(5)
Clearing (3) of fractions,	$2x + 3z = 657$	(6)
Multiplying (4) by 3,	$21x + 9y = 4599$	(7)
Subtracting (5) from (7),	$21x - 5z = 2628$	(8)
Multiplying (8) by 3,	$63x - 15z = 7884$	(9)
Multiplying (6) by 5,	$10x + 15z = 3285$	(10)
Adding (9) and (10),	$73x = 11169$	(11)
Whence,	$x = 153$	(12)
Substituting in (3),	$z + 102 = 219$	(13)
Whence,	$z = 117$	(14)
Substituting in (2),	$y + 65 = 219$	(15)
Whence,	$y = 154$	(16)

14. Let

x = the first number,

y = the second number,

z = the third number,

and

u = the fourth number.

Then,

$$x + \frac{y}{2} = 357 \quad (1)$$

$$y + \frac{z}{3} = 476 \quad (2)$$

$$z + \frac{u}{4} = 595 \quad (3)$$

and

$$u + \frac{x}{5} = 714 \quad (4)$$

Clearing (1) of fractions,

$$2x + y = 714 \quad (5)$$

Clearing (2) of fractions,

$$3y + z = 1428 \quad (6)$$

Clearing (3) of fractions,

$$4z + u = 2380 \quad (7)$$

Clearing (4) of fractions,

$$5u + x = 3570 \quad (8)$$

Multiplying (7) by 5,

$$20z + 5u = 11900 \quad (9)$$

Subtracting (8) from (9),

$$20z - x = 8330 \quad (10)$$

Multiplying (6) by 20,

$$60y + 20z = 28560 \quad (11)$$

Subtracting (10) from (11),

$$x + 60y = 20230 \quad (12)$$

Multiplying (5) by 60,

$$120x + 60y = 42840 \quad (13)$$

Subtracting (12) from (13),

$$119x = 22610 \quad (14)$$

Whence,

$$x = 190 \quad (15)$$

Substituting in (4),

$$u + 38 = 714 \quad (16)$$

Whence,

$$u = 676 \quad (17)$$

Substituting in (3),

$$z + 169 = 595 \quad (18)$$

Whence,

$$z = 426 \quad (19)$$

Substituting in (2),

$$y + 142 = 476 \quad (20)$$

Whence,

$$y = 334 \quad (21)$$

15. Let

x = price first quality,

y = price second quality,

and

z = price third quality.

Then, $3x + 4y + 2z = 60$ (1)

$4x + y + 5z = 59$ (2)

and $x + 10y + 3z = 90$ (3)

Multiplying (3) by 4, $4x + 40y + 12z = 360$ (4)

Subtracting (2) from (4), $39y + 7z = 301$ (5)

Multiplying (3) by 3, $3x + 30y + 9z = 270$ (6)

Subtracting (1) from (6), $26y + 7z = 210$ (7)

Subtracting (7) from (5), $13y = 91$ (8)

Whence, $y = 7$ (9)

Substituting in (7), $182 + 7z = 210$ (10)

Transposing and uniting, $7z = 28$ (11)

Whence, $z = 4$ (12)

Substituting in (3), $x + 70 + 12 = 90$ (13)

Whence, $x = 8$ (14)

16. Let

$x =$ no. A's squirrels,

$y =$ no. B's squirrels,

and

$z =$ no. C's squirrels.

$x - y - z =$ A's after first change,

$2y =$ B's after first change,

$2z =$ C's after first change.

$2x - 2y - 2z =$ A's after second change,

$2y - (x - y - z) - 2z = -x + 3y - z =$ B's aft. 2d change,

$4z =$ C's after second change.

$4x - 4y - 4z =$ A's after third change,

$-2x + 6y - 2z =$ B's after third change,

$4z - (2x - 2y - 2z) - (-x + 3y - z) = -x - y + 7z$
 $=$ C's after third change.

From the conditions, $x + y + z = 96$ (1)

$4x - 4y - 4z = -2x + 6y - 2z$ (2)

and $-2x + 6y - 2z = -x - y + 7z$ (3)

Transp. and uniting in (2), $6x - 10y - 2z = 0$ (4)

Dividing (4) by 2, $3x - 5y - z = 0$ (5)

Transp. and uniting in (3),	$-x + 7y - 9z = 0$	(6)
Adding (1) and (6),	$8y - 8z = 96$	(7)
Dividing (7) by 8,	$y - z = 12$	(8)
Multiplying (1) by 3,	$3x + 3y + 3z = 288$	(9)
Subtracting (5) from (9),	$8y + 4z = 288$	(10)
Dividing (10) by 4,	$2y + z = 72$	(11)
Adding (8) and (11),	$3y = 84$	(12)
Whence,	$y = 28$	(13)
Substituting in (11),	$56 + z = 72$	(14)
Whence,	$z = 16$	(15)
Substituting in (1),	$x + 28 + 16 = 96$	(16)
Whence,	$x = 52$	(17)

NOTE. As the whole number of squirrels remains the same, the student can test the accuracy of his statement by observing whether the sum of A's, B's, and C's after each change is $x + y + z$.

It will be observed that at each change two of the men double theirs, while the share of the third is diminished by each of the others.

INTERPRETATION OF NEGATIVE RESULTS.

(ART. 180, pp. 161-163.)

3. Let	$x = \text{the number.}$
Then,	$10 - x = 15$
Whence,	$-x = 5$
Or,	$x = -5$

The negative result indicates that the problem should read: "What number must be *added* to 10, that the *sum* shall be 15?" The answer to the problem in this form will be 5.

4. Let	$x = \text{the number.}$
Then,	$\frac{x}{5} - \frac{x}{4} = 4$
Clearing of fractions,	$4x - 5x = 80$

Whence,

$$-x = 80$$

Or,

$$x = -80$$

This result indicates that 80 is the answer to the problem, "What number is that whose *fourth part exceeds its fifth part by 4?*"

5. Let

 $x =$ number of years to elapse.

Then,

$$40 + x : 36 + x :: 6 : 5$$

Mult. extremes and means, $200 + 5x = 216 + 6x$

Whence,

$$-x = 16$$

Or,

$$x = -16$$

This result indicates that their ages were as 6 to 5 16 years *before*, instead of after, their marriage; that is, when one was 24 years old, and the other 20. If, then, the latter part of the problem were so modified as to read: "How many years before their marriage was his age to hers as 6 to 5?" the proportion would be

$$40 - x : 36 - x :: 6 : 5$$

$$\text{Whence, } x = 16$$

6. Let

 $x =$ number of rods to be added.

Then,

$$5(8 + x) = 30$$

Developing,

$$40 + 5x = 30$$

Transposing and uniting,

$$5x = -10$$

Whence,

$$x = -2$$

The 2 rods are, then, to be *subtracted* from the length, instead of added to it; and the enunciation must be modified accordingly, in order to give a positive result.

7. Let

 $x =$ the number.

Then,

$$\frac{x}{3} + \frac{x}{5} - 7 = x$$

Clearing of fractions,

$$5x + 3x - 105 = 15x$$

Transposing and uniting,

$$-7x = 105$$

Dividing by -7 ,

$$x = -15$$

In order that the result may be $+15$, the problem should read *increased by 7*, instead of "*diminished by 7*."

8. Let

 $x =$ the numerator,

and

 $y =$ the denominator.

Then,

$$\frac{x+2}{y} = \frac{1}{4} \quad (1)$$

and

$$\frac{x}{y+2} = \frac{1}{2} \quad (2)$$

Clearing (1) of fractions,

$$4x + 8 = y \quad (3)$$

Clearing (2) of fractions,

$$2x = y + 2 \quad (4)$$

Subtracting (4) from (3),

$$2x + 8 = -2 \quad (5)$$

Transposing and uniting,

$$2x = -10 \quad (6)$$

Whence,

$$x = -5 \quad (7)$$

Substituting in (3),

$$y = -20 + 8 = -12 \quad (8)$$

Therefore,

$$\frac{x}{y} = \frac{-5}{-12} \quad (9)$$

This result indicates that the fraction $\frac{5}{12}$ is such that 2 *subtracted* from its numerator gives $\frac{1}{12}$, and 2 *subtracted* from its denominator gives $\frac{1}{12}$.

9. Let

 $x =$ number of years.

Then,

$$15 + x = \frac{45 + x}{4}$$

Clearing of fractions,

$$60 + 4x = 45 + x$$

Transposing and uniting,

$$3x = -15$$

Whence,

$$x = -5$$

Hence the son was one fourth of the age of the father 5 years ago.

10. Let

 $x =$ the father's wages,

and

 $y =$ the son's wages.

$$\text{Then,} \quad 12x + 8y = 22 \quad (1)$$

$$\text{and} \quad 10x + 4y = 19 \quad (2)$$

$$\text{Dividing (1) by 2,} \quad 6x + 4y = 11 \quad (3)$$

$$\text{Subtracting (3) from (2),} \quad 4x = 8 \quad (4)$$

$$\text{Whence,} \quad x = 2 \quad (5)$$

$$\text{Substituting in (3),} \quad 12 + 4y = 11 \quad (6)$$

$$\text{Transposing and uniting,} \quad 4y = -1 \quad (7)$$

$$\text{Whence,} \quad y = -\frac{1}{4} \quad (8)$$

The negative result indicates that the father *paid out* money for his son, instead of receiving wages for his labor.

ZERO AND INFINITY.

(ART. 184, pp. 165, 166.)

2. Let x = number of years.

$$\text{Then,} \quad x + 40 = x + 60$$

$$\text{Hence,} \quad x - x = 20$$

$$\text{Or,} \quad 0x = 20$$

$$\text{Whence,} \quad x = \frac{20}{0} = \infty$$

This result shows that the problem is impossible; that is, that A and B can never be of the same age.

3. Let x = no. bought in first flock,
and y = no. bought in second flock.

$$\text{Then} \quad \frac{3x}{2} = \text{cost of first flock,}$$

$$\text{and} \quad 2y = \text{cost of second flock,}$$

$$2(x - 30) = \text{am't rec'd for first flock,}$$

$$\frac{1}{2}(y - 56) = \text{am't rec'd for second flock.}$$

$$\text{From the conditions,} \quad x + y = 400 \quad (1)$$

$$\text{Also,} \quad 2(x - 30) + \frac{1}{2}(y - 56) = \frac{3x}{2} + 2y \quad (2)$$

Clearing (2) of fractions,

$$4(x - 30) + 5(y - 56) = 3x + 4y \quad (3)$$

Expanding, $4x - 120 + 5y - 280 = 3x + 4y \quad (4)$

Transposing and uniting in (4), $x + y = 400 \quad (5)$

Equation (1), $x + y = 400$

Subtracting (1) from (5), $0y = 0 \quad (6)$

Whence, $y = \frac{0}{0} \quad (7)$

Substituting in (1), $x + \frac{0}{0} = 400 \quad (8)$

Whence, $x = 400 - \frac{0}{0} \quad (9)$

From the fact that $\frac{0}{0}$ is a symbol of indetermination, we infer that each flock may contain any finite number whatever. There are, however, certain limits, beyond which the numbers cannot pass, without involving *negative* results.

1. When any number is assumed for one flock, the other becomes determinate; for the sum of the two must be 400.

2. It is evident, from the original form of equation (2), that the first flock must contain more than 30, and the second more than 56.

3. It appears from 1 and 2 that the first flock must contain less than 344, and the second less than 370.

Hence the number in the first flock is from 31 to 343, and the number in the second is from 369 to 57; that is, if the first is 31, the second is 369, if the first is 32, the second is 368, and so on, thus giving 313 different sets of numbers which will answer the conditions.

NOTE. That the problem is indeterminate is readily ascertained, without obtaining the result $\frac{0}{0}$; for the two equations, though very different in their original forms, are not *independent* (Arts. 168, 169), since they finally assume the same form, $x + y = 400$.

INVOLUTION.

(ART. 190, pp. 168–170.)

$$7. \text{ Ans. } 243 a^{10} x^{10}. \quad | \quad 15. \text{ Ans. } 16 a^4 x^8.$$

(ART. 191, pp. 170, 171.)

$$8. \text{ Ans. } \frac{a^{10} c^5 a^{15}}{32 b^{10}}.$$

(ART. 192, pp. 171, 172.)

In these examples, as well as the preceding ones, there are three points to be considered: the *sign* of the power, which is determined by Arts. 187, 188, the *coefficient*, and the *exponents*, both of which are determined by the rule found in Art. 190. The signs of the exponents must be determined by the rules for ordinary multiplication.

It has already been shown that the negative exponent expresses an unperformed division, and that a quantity containing a negative exponent is, in reality, the same as a fraction.

3. The answer to this example cannot be both $+$ and $-$ for the same value of n ; but if n is *even*, it is $+$, while if n is *odd*, it is $-$.

5. Multiplying the exponents by -2 changes their signs.

6. The answer is negative because the original quantity is negative and the exponent -3 is an odd number.

7. The result is positive because the exponent -4 is an even number. The coefficient becomes

$$2^{-4} = \frac{1}{2^4} = \frac{1}{16}.$$

8. If the negative exponent is removed, the answer becomes $\frac{c^{6n}}{a^{6n} b^{6m}}$.

The negative exponents are also readily removed in examples 9 and 10, by transferring factors according to Art. 126.

POWERS OF BINOMIALS.

(ART. 200, pp. 176-178.)

In writing any power of a binomial, the student has only to apply the principles found in Arts. 196-200.

4. Ans. $a^4 + 4 a^3 y + 6 a^2 y^2 + 4 a y^3 + y^4$.

6. The coefficients of the seventh power are thus obtained : 1, 7, $\frac{7 \times 6}{2} = 21$, $\frac{21 \times 5}{3} = 35$, $\frac{35 \times 4}{4} = 35$, $\frac{35 \times 3}{5} = 21$, &c.

7. The first form of the answer, according to the Binomial Theorem, would be $a^7 + 3 a^6 1 + 3 a^5 1^2 + 1^3$.

8. $(1-x)^6 = (1)^6 - 6(1)^5 x + 15(1)^4 x^2 - 20(1)^3 x^3 + 15(1)^2 x^4 - 6(1)x^5 + x^6$. The powers of 1, being 1, have no effect as factors, and can therefore be omitted.

9. The coefficients of the eighth power are thus obtained : 1, 8, $\frac{8 \times 7}{2} = 28$, $\frac{28 \times 6}{3} = 56$, $\frac{56 \times 5}{4} = 70$, $\frac{70 \times 4}{5} = 56$, &c.

(ART. 201, pp. 178, 179.)

2. $(3a + 2b)^4 = (3a)^4 + 4(3a)^3(2b) + 6(3a)^2(2b)^2 + 4(3a)(2b)^3 + (2b)^4 = 81a^4 + 216a^3b + 216a^2b^2 + 96ab^3 + 16b^4$.

$$3. \quad (2a-3x)^3 = (2a)^3 - 3(2a)^2(3x) + 3(2a)(3x)^2 - (3x)^3 \\ = 8a^3 - 36a^2x + 54ax^2 - 27x^3.$$

$$4. \quad (1+3x)^4 = (1)^4 + 4(1)^3(3x) + 6(1)^2(3x)^2 \\ + 4(1)(3x)^3 + (3x)^4 = 1 + 12x + 54x^2 + 108x^3 + 81x^4.$$

$$5. \quad (a^2+b^2)^3 = (a^2)^3 + 3(a^2)^2(b^2) + 3(a^2)(b^2)^2 + (b^2)^3 \\ = a^6 + 3a^4b^2 + 3a^2b^4 + b^6.$$

$$6. \quad (3x-5)^3 = (3x)^3 - 3(3x)^2 \cdot 5 + 3(3x) \cdot 5^2 - 5^3 \\ = 27x^3 - 135x^2 + 225x - 125, \text{ Ans.}$$

$$7. \quad (3xy-a)^2 = (3xy)^2 - 2(3xy)a + a^2 \\ = 9x^2y^2 - 6axy + a^2.$$

$$8. \quad (\frac{1}{2}ab+c)^2 = (\frac{1}{2}ab)^2 + 2(\frac{1}{2}ab)c + c^2 \\ = \frac{1}{4}a^2b^2 + abc + c^2.$$

$$9. \quad (x-\frac{p}{2})^2 = x^2 - 2x(\frac{p}{2}) + (\frac{p}{2})^2 = x^2 - px + \frac{p^2}{4}.$$

$$10. \quad (3x+\frac{1}{x})^2 = (3x)^2 + 2(3x)(\frac{1}{x}) + (\frac{1}{x})^2 = 9x^2 + 6 + \frac{1}{x^2}.$$

$$11. \quad (a+a^{-1})^3 = a^3 + 3a^2(a^{-1}) + 3a(a^{-1})^2 + (a^{-1})^3 \\ = a^3 + 3a + 3a^{-1} + a^{-3}.$$

NOTE. It will be observed that negative exponents might be employed in the 10th, or the fractional form in the 11th.

$$12. \quad (x^2+3y^2)^5 = (x^2)^5 + 5(x^2)^4(3y^2) + 10(x^2)^3(3y^2)^2 \\ + 10(x^2)^2(3y^2)^3 + 5(x^2)(3y^2)^4 + (3y^2)^5 = x^{10} + 15x^8y^2 \\ + 90x^6y^4 + 270x^4y^6 + 405x^2y^8 + 243y^{10}.$$

$$13. \quad (\frac{1}{2}x - \frac{2}{3}y)^3 = (\frac{1}{2}x)^3 - 3(\frac{1}{2}x)^2(\frac{2}{3}y) + 3(\frac{1}{2}x)(\frac{2}{3}y)^2 \\ - (\frac{2}{3}y)^3 = \frac{1}{8}x^3 - \frac{1}{2}x^2y + \frac{2}{3}xy^2 - \frac{8}{27}y^3.$$

$$14. \quad (x^2-2x)^6 = (x^2)^6 - 6(x^2)^5(2x) + 15(x^2)^4(2x)^2 \\ - 20(x^2)^3(2x)^3 + 15(x^2)^2(2x)^4 - 6(x^2)(2x)^5 + (2x)^6 \\ = x^{12} - 12x^{11} + 60x^{10} - 160x^9 + 240x^8 - 192x^7 + 64x^6.$$

EVOLUTION.

SQUARE ROOT OF NUMBERS.

(ART. 215, pp. 183-185.)

$$\begin{array}{r}
 \text{(3.)} \\
 \begin{array}{r}
 6\dot{1}1\dot{5}2\dot{4} \mid 782 \\
 49 \\
 \hline
 148 \mid \begin{array}{r} 1215 \\ 1184 \end{array} \\
 \hline
 1562 \mid \begin{array}{r} 3124 \\ 3124 \end{array}
 \end{array}
 \end{array}$$

$$\begin{array}{r}
 \text{(4.)} \\
 \begin{array}{r}
 5\dot{6}\dot{6}4\dot{4} \mid 238 \\
 4 \\
 \hline
 43 \mid \begin{array}{r} 166 \\ 129 \end{array} \\
 \hline
 468 \mid \begin{array}{r} 3744 \\ 3744 \end{array}
 \end{array}
 \end{array}$$

$$\begin{array}{r}
 \text{(5.)} \\
 \begin{array}{r}
 6\dot{5}6\dot{1} \mid 81, \text{Ans.} \\
 64 \\
 \hline
 161 \mid \begin{array}{r} 161 \\ 161 \end{array}
 \end{array}
 \end{array}$$

$$\begin{array}{r}
 \text{(6.)} \\
 \begin{array}{r}
 2\dot{1}1\dot{6} \mid 46 \\
 16 \\
 \hline
 86 \mid \begin{array}{r} 516 \\ 516 \end{array}
 \end{array}
 \end{array}$$

$$\begin{array}{r}
 \text{(7.)} \\
 \begin{array}{r}
 1\dot{0}2\dot{4}6\dot{4}0\dot{1} \mid 3201 \\
 9 \\
 \hline
 62 \mid \begin{array}{r} 124 \\ 124 \end{array} \\
 \hline
 6401 \mid \begin{array}{r} 6401 \\ 6401 \end{array}
 \end{array}
 \end{array}$$

$$\begin{array}{r}
 \text{(8.)} \\
 \begin{array}{r}
 1\dot{6}2\dot{4}0\dot{9} \mid 4.03 \\
 16 \\
 \hline
 803 \mid \begin{array}{r} 2409 \\ 2409 \end{array}
 \end{array}
 \end{array}$$

$$\begin{array}{r}
 \text{(9.)} \\
 \begin{array}{r}
 .9\dot{4}0\dot{9} \mid .97 \\
 81 \\
 \hline
 187 \mid \begin{array}{r} 1309 \\ 1309 \end{array}
 \end{array}
 \end{array}$$

$$\begin{array}{r}
 \text{(10.)} \\
 \begin{array}{r}
 .0\dot{0}8\dot{1} \mid .09 \\
 81 \\
 \hline
 \end{array}
 \end{array}$$

(11.)

$$\begin{array}{r}
 .00\dot{6}0\dot{0}0\dot{0}0\dot{0} \quad | .0774+ \\
 \underline{49} \\
 147 \quad | 1100 \\
 \quad \quad | 1029 \\
 \hline
 1544 \quad | 7100 \\
 \quad \quad | 6176
 \end{array}$$

(12.)

$$\begin{array}{r}
 12.0\dot{0}0\dot{0}0\dot{0}0\dot{0}0\dot{0}0\dot{0} \quad | 3.464101+ \\
 \underline{9} \\
 64 \quad | 300 \\
 \quad \quad | 256 \\
 \hline
 686 \quad | 4400 \\
 \quad \quad | 4116 \\
 \hline
 6924 \quad | 28400 \\
 \quad \quad | 27696 \\
 \hline
 69281 \quad | 70400 \\
 \quad \quad | 69281 \\
 \hline
 6928201 \quad | 11190000 \\
 \quad \quad | 6928201
 \end{array}$$

(13.)

$$\begin{array}{r}
 .0\dot{0}0\dot{0}0\dot{0}12\dot{3}2\dot{1} \quad | .00111 \\
 \underline{1} \\
 21 \quad | 23 \\
 \quad \quad | 21 \\
 \hline
 221 \quad | 221 \\
 \quad \quad | 221
 \end{array}$$

(ART. 216, pp. 185, 186.)

$$2. \quad \sqrt{\frac{16}{169}} = \frac{\sqrt{16}}{\sqrt{169}} = \frac{4}{13}.$$

$$3. \quad \sqrt{\frac{484}{729}} = \frac{\sqrt{484}}{\sqrt{729}} = \frac{22}{27}.$$

$$4. \quad \sqrt{2\frac{47}{121}} = \sqrt{\frac{289}{121}} = \frac{\sqrt{289}}{\sqrt{121}} = \frac{17}{11} = 1\frac{6}{11}.$$

$$5. \sqrt{\frac{8}{18}} = \sqrt{\frac{4}{9}} = \frac{\sqrt{4}}{\sqrt{9}} = \frac{2}{3}.$$

$$6. \sqrt{\frac{1899}{10339}} = \sqrt{\frac{1899 \div 211}{10339 \div 211}} = \sqrt{\frac{9}{49}} = \frac{\sqrt{9}}{\sqrt{49}} = \frac{3}{7}.$$

(7.)

$$\frac{9}{13} = .692308, \text{ nearly.}$$

$$\begin{array}{r} .692308 \quad .832+ \\ 64 \\ \hline 163 \overline{) 523} \\ \underline{489} \\ 1662 \overline{) 3408} \\ \underline{3324} \end{array}$$

(8.)

$$\frac{16}{1031} = .01551891 +$$

$$\begin{array}{r} .01551891 \quad .1246, \text{ nearly.} \\ 1 \\ \hline 22 \overline{) 55} \\ \underline{44} \\ 244 \overline{) 1118} \\ \underline{976} \\ 2486 \overline{) 14291} \\ \underline{14916} \end{array}$$

CUBE ROOT OF NUMBERS.

(ART. 222, pp. 187-190.)

(3.)

$$\begin{array}{r} 941192 \overline{) 98} \\ 729 \\ \hline 212192 \\ \hline 212192 \end{array}$$

$$\begin{array}{r} 24300 \\ 2160 \\ 64 \\ \hline 26524 \times 8 = \end{array}$$

(4.)

$$\begin{array}{r} 389017 \overline{) 73} \\ 343 \\ \hline 46017 \\ \hline 46017 \end{array}$$

$$\begin{array}{r} 14700 \\ 630 \\ 9 \\ \hline 15339 \times 3 = \end{array}$$

(5.)

$$\begin{array}{r} 37259704 \overline{) 334} \\ 27 \\ \hline 10259 \\ \hline 8937 \\ \hline 1322704 \\ \hline 1322704 \end{array}$$

$$\begin{array}{r} 2700 \\ 270 \\ 9 \\ \hline 2979 \times 3 = \\ 326700 \\ 3960 \\ 16 \\ \hline 330676 \times 4 = \end{array}$$

(6.)

251239591	631
216	
35239	
34047	
1192591	
1192591	

10800
 540
 9

 11349

$\times 3 =$

1190700
 1890
 1

 1192591

(7.)

46268279	359*
27	
19268	
15875	
3393279	
3393279	

2700
 450
 25

 3175

$\times 5 =$

367500
 9450
 81

 377031

(8.)

1481544	114
1	
481	
331	
150544	
150544	

300
 30
 1

 331

$\times 1 =$

36300
 1320
 16

 37636

* It is found, by trial, that 7 and 6 are too large, although 27 is contained in 192 seven times.

$$\begin{array}{r}
 \text{(9.)} \\
 .008649000000 \quad .2052+ \\
 \hline
 8 \\
 \hline
 \begin{array}{r}
 120000 \\
 3000 \\
 25 \\
 \hline
 123025 \times 5 = 615125 \\
 12607500 \\
 12300 \\
 4 \\
 \hline
 12619804 \times 2 = 25239608
 \end{array}
 \end{array}$$

(ART. 223, pp. 190, 191.)

$$2. \sqrt[3]{\frac{125}{343}} = \frac{\sqrt[3]{125}}{\sqrt[3]{343}} = \frac{5}{7}.$$

$$\begin{array}{r}
 3. \quad \begin{array}{r} 68921 \\ 64 \end{array} \bigg| 41. \quad \begin{array}{r} 59319 \\ 27 \end{array} \bigg| 39 \\
 \begin{array}{r} 4800 \\ 120 \\ 1 \\ \hline 4921 \times 1 = 4921 \end{array} \quad \begin{array}{r} 2700 \\ 810 \\ 81 \\ \hline 3591 \times 9 = 32319 \end{array}
 \end{array}$$

$$\sqrt[3]{\frac{68921}{59319}} = \frac{\sqrt[3]{68921}}{\sqrt[3]{59319}} = \frac{41}{39} = 1\frac{2}{39}.$$

(4.)

$$\frac{2}{3} = .105263158, \text{ nearly.}$$

$$\begin{array}{r}
 .105263158 \quad .472+ \\
 \hline
 64 \\
 \hline
 \begin{array}{r}
 4800 \\
 840 \\
 49 \\
 \hline
 5689 \times 7 = 39823 \\
 662700 \\
 2820 \\
 4 \\
 \hline
 665524 \times 2 = 1331048
 \end{array}
 \end{array}$$

* It is found, by trial, that 8 is too large, although 48 is contained in 412 eight times.

ROOTS OF MONOMIALS.

(ART. 224, pp. 191-193.)

6. Ans. $12 a^2 b c^3$.

It will be a valuable exercise for the student to express the answers in various ways, whenever they are capable of it. Thus,

$$9. \pm \sqrt[3]{\frac{2b x^3}{3a^2 y}} = \pm \sqrt[3]{\frac{2}{3} a^{-2} b x^3 y^{-1}} = \pm 2 \times 3^{-1} a^{-2} b x^3 y^{-1}.$$

$$10. \sqrt[3]{\frac{27x^3}{729y^3}} = \sqrt[3]{\frac{x^3}{27y^3}} = \frac{x}{3y} = \frac{1}{3} x y^{-1} = 3^{-1} x y^{-1}.$$

$$12. -9 a^{-1} b^{-2} = -\frac{9}{ab^2}.$$

14. We here *divide* each exponent by the *index* of the root, which is 2, or *multiply* each by the *fractional exponent*, which is $\frac{1}{2}$. $\pm 13 a^{\frac{3}{2}} b^{-\frac{1}{2}} c^{-1} = \pm \frac{13 a^{\frac{3}{2}}}{b^{\frac{1}{2}} c}.$

$$15. \frac{a b^{\frac{1}{2}}}{2 x^{\frac{1}{2}} y} = \frac{1}{2} a b^{\frac{1}{2}} x^{-\frac{1}{2}} y^{-1}.$$

$$16. a b^{\frac{m}{n}} c^2 d^{-3} = \frac{a b^{\frac{m}{n}} c^2}{d^3}.$$

SQUARE ROOT OF POLYNOMIALS.

(ART. 225, pp. 193-196.)

(4.)

$$\begin{array}{r|l} a^4 + 4a^2b + 4b^2 & a^2 + 2b \\ a^4 & \\ \hline 2a^2 + 2b & 4a^2b + 4b^2 \\ & 4a^2b + 4b^2 \end{array}$$

(5.)

$$\begin{array}{r}
 9x^4 - 12x^3 + 16x^2 - 8x + 4 \quad | \quad 3x^2 - 2x + 2 \\
 9x^4 \\
 \hline
 6x^2 - 2x \quad | \quad -12x^3 + 16x^2 \\
 \quad \quad \quad | \quad -12x^3 + 4x^2 \\
 \hline
 6x^2 - 4x + 2 \quad | \quad 12x^2 - 8x + 4 \\
 \quad \quad \quad | \quad 12x^2 - 8x + 4 \\
 \hline
 \end{array}$$

(6.)

$$\begin{array}{r}
 x^2 + 4bx + 4b^2 \quad | \quad x + 2b \\
 x^2 \\
 \hline
 2x + 2b \quad | \quad 4bx + 4b^2 \\
 \quad \quad \quad | \quad 4bx + 4b^2 \\
 \hline
 \end{array}
 \quad \text{Ans. } x + 2b.$$

(7.)

$$\begin{array}{r}
 a^4 + 4a^3b + 10a^2b^2 + 12ab^3 + 9b^4 \quad | \quad a^2 + 2ab + 3b^2 \\
 a^4 \\
 \hline
 2a^2 + 2ab \quad | \quad 4a^3b + 10a^2b^2 \\
 \quad \quad \quad | \quad 4a^3b + 4a^2b^2 \\
 \hline
 2a^2 + 4ab + 3b^2 \quad | \quad 6a^2b^2 + 12ab^3 + 9b^4 \\
 \quad \quad \quad | \quad 6a^2b^2 + 12ab^3 + 9b^4 \\
 \hline
 \end{array}$$

(8.)

$$\begin{array}{r}
 a^4 - 2a^3 + 2a^2 - a + \frac{1}{4} \quad | \quad a^2 - a + \frac{1}{4} \\
 a^4 \\
 \hline
 2a^2 - a \quad | \quad -2a^3 + 2a^2 \\
 \quad \quad \quad | \quad -2a^3 + a^2 \\
 \hline
 2a^2 - 2a + \frac{1}{2} \quad | \quad a^2 - a + \frac{1}{4} \\
 \quad \quad \quad | \quad a^2 - a + \frac{1}{4} \\
 \hline
 \end{array}$$

(9.)

$$\begin{array}{r}
 x^4 - 2x^2 + 1 \quad | \quad x^2 - 1 \\
 x^4 \\
 \hline
 2x^2 - 1 \quad | \quad -2x^2 + 1 \\
 \quad \quad \quad | \quad -2x^2 + 1 \\
 \hline
 \end{array}$$

(10.)

$$\begin{array}{r}
 a^2 - 2 + a^{-2} \quad | \quad a - a^{-1} \\
 a^2 \\
 \hline
 2a - a^{-1} \quad | \quad -2 + a^{-2} \\
 \quad \quad \quad | \quad -2 + a^{-2} \\
 \hline
 \end{array}$$

(11.)

$$\begin{array}{r}
 4a^2 - 12ab + 9b^2 + 4ax - 6bx + x^2 \quad | \quad 2a - 3b + x \\
 \underline{4a^2} \\
 4a - 3b \quad | \quad -12ab + 9b^2 \\
 \quad | \quad \underline{-12ab + 9b^2} \\
 \quad | \quad 4a - 6b + x \quad | \quad 4ax - 6bx + x^2 \\
 \quad | \quad \quad | \quad \underline{4ax - 6bx + x^2}
 \end{array}$$

(12.)

$$\begin{array}{r}
 a^4 + 8a^3b + 24a^2b^2 + 32ab^3 + 16b^4 \quad | \quad a^2 + 4ab + 4b^2 \\
 \underline{a^4} \\
 2a^2 + 4ab \quad | \quad 8a^3b + 24a^2b^2 \\
 \quad | \quad \underline{8a^3b + 16a^2b^2} \\
 2a^2 + 8ab + 4b^2 \quad | \quad 8a^3b^2 + 32a^2b^3 + 16b^4 \\
 \quad | \quad \underline{8a^3b^2 + 32a^2b^3 + 16b^4} \\
 \quad | \quad a^2 + 4ab + 4b^2 \quad | \quad a + 2b \\
 \quad | \quad \underline{a^2} \\
 2a + 2b \quad | \quad 4ab + 4b^2 \\
 \quad | \quad \underline{4ab + 4b^2}
 \end{array}$$

CUBE ROOT OF POLYNOMIALS.

(ART. 226, pp. 196-200.)

(4.)

$$\begin{array}{r}
 x^3 + 3x^2y + 3xy^2 + y^3 \quad | \quad x + y \\
 \underline{x^3} \\
 3x^2 + 3xy + y^2 \quad | \quad 3x^2y + 3xy^2 + y^3 \\
 \quad | \quad \underline{3x^2y + 3xy^2 + y^3}
 \end{array}$$

(5.)

$$\begin{array}{r}
 y^6 - 3y^5 + 5y^3 - 3y - 1 \quad | \quad y^2 - y - 1 \\
 \underline{y^6} \\
 3y^4 - 3y^3 + y^2 \quad | \quad -3y^5 + 5y^3 \\
 \quad | \quad \underline{-3y^5 + 3y^4 - y^3} \\
 3y^4 - 6y^3 + 3y + 1 \quad | \quad -3y^4 + 6y^3 - 3y - 1 \\
 \quad | \quad \underline{-3y^4 + 6y^3 - 3y - 1}
 \end{array}$$

NOTE. In the last complete divisor, $3y^3$ and $-3y^3$ cancel each other.

$$\begin{array}{r}
 \text{(6.)} \\
 \begin{array}{r}
 27x^2 + 54x^2y + 36xy^2 + 8y^3 \\
 27x^2 \\
 \hline
 27x^2 + 18xy + 4y^2
 \end{array}
 \begin{array}{r}
 3x + 2y \\
 \hline
 54x^2y + 36xy^2 + 8y^3 \\
 54x^2y + 36xy^2 + 8y^3 \\
 \hline
 \end{array}
 \end{array}$$

Ans. $3x + 2y$.

7. It will be observed that the solution of this example is precisely the same as that of the 3d, except that m is substituted for x .

$$\begin{array}{r}
 \text{(8.)} \\
 \begin{array}{r}
 a^3 + 3a + 3a^{-1} + a^{-3} \\
 a^2 \\
 \hline
 3a^2 + 3 + a^{-2}
 \end{array}
 \begin{array}{r}
 a + a^{-1} \\
 \hline
 3a + 3a^{-1} + a^{-3} \\
 3a + 3a^{-1} + a^{-3} \\
 \hline
 \end{array}
 \end{array}$$

RADICALS.

REDUCTION OF RADICALS.

(ART. 233, pp. 201, 202.)

2. $\sqrt{9a^4x} = \sqrt{9a^4} \times \sqrt{x} = 3a^2\sqrt{x}$.
3. $\sqrt{32a^2x} = \sqrt{16a^2} \times \sqrt{2x} = 4a\sqrt{2x}$.
4. $7\sqrt{80x} = 7\sqrt{16 \times 5x} = 7 \times 4\sqrt{5x} = 28\sqrt{5x}$.
5. $a\sqrt{125b^3} = a\sqrt{25b^2 \times 5b} = a \times 5b\sqrt{5b} = 5ab\sqrt{5b}$.
6. $\sqrt[3]{64a^6b^3} = \sqrt[3]{64a^3b^3} \times \sqrt[3]{a^3} = 4ab\sqrt[3]{a^3}$.
7. $\sqrt{50a^2b^2c^2} = \sqrt{25b^2c^2 \times 2a} = 5bc\sqrt{2a}$, Ans.
8. $(ax^3 + bx^4)^{\frac{1}{3}} = \sqrt[3]{x^3(ax + bx^2)} = x(ax + bx^2)^{\frac{1}{3}}$.
9. $2(x^3 - a^2x^2)^{\frac{1}{2}} = 2\sqrt{x^2(x - a^2)} = 2x(x - a^2)^{\frac{1}{2}}$.
10. $\sqrt[3]{5(a^3 + a^4b)} = \sqrt[3]{5a^3(1 + ab)} = a\sqrt[3]{5(1 + ab)}$.

$$11. \quad 6\sqrt{54a^3b^3c} = 6\sqrt{9a^2b^2 \times 6abc} = 6 \times 3ab\sqrt{6abc} \\ = 18ab\sqrt{6abc}.$$

$$12. \quad 3\sqrt[5]{32a^5b^5c^5} = 3\sqrt[5]{32a^5c^5 \times ab^2} = 3 \times 2ac\sqrt[5]{ab^2} \\ = 6ac\sqrt[5]{ab^2}.$$

$$13. \quad (72x + 108y)^{\frac{1}{2}} = \sqrt{36(2x + 3y)} = 6(2x + 3y)^{\frac{1}{2}}.$$

$$14. \quad 5(a-b)\sqrt{a^2c + 2abc + b^2c} = 5(a-b)\sqrt{c(a+b)^2} \\ = 5(a-b)(a+b)\sqrt{c} = 5(a^2 - b^2)\sqrt{c}.$$

(ART. 234, p. 203.)

$$3. \quad 2\sqrt{\frac{3}{8}} = 2\sqrt{\frac{3}{8} \times \frac{2}{2}} = 2\sqrt{\frac{6}{16}} = 2\sqrt{\frac{1}{16} \times 6} = 2 \times \frac{1}{4}\sqrt{6} \\ = \frac{2}{4}\sqrt{6} = \frac{1}{2}\sqrt{6}.$$

$$4. \quad \frac{3}{4}\sqrt{\frac{2ax^3}{9}} = \frac{3}{4}\sqrt{\frac{2ax^3}{9} \times \frac{3}{3}} = \frac{3}{4}\sqrt{\frac{6ax^3}{9}} = \frac{3}{4}\sqrt{\frac{x^3}{9} \times 6ax} \\ = \frac{3}{4} \times \frac{x}{3}\sqrt{6ax} = \frac{x}{4}\sqrt{6ax}.$$

$$5. \quad \left(\frac{a^4b^3}{4}\right)^{\frac{1}{3}} = \left(\frac{a^4b^3}{4} \times \frac{2}{2}\right)^{\frac{1}{3}} = \left(\frac{2a^4b^3}{8}\right)^{\frac{1}{3}} = \left(\frac{a^3b^3}{8} \times 2a\right)^{\frac{1}{3}} \\ = \frac{ab}{2}(2a)^{\frac{1}{3}}.$$

$$6. \quad 4\sqrt{\frac{3a^2}{8b}} = 4\sqrt{\frac{3a^2}{8b} \times \frac{2b}{2b}} = 4\sqrt{\frac{6a^2b}{16b^2}} = 4\sqrt{\frac{a^2}{16b^2} \times 6ab} \\ = 4 \times \frac{a}{4b}\sqrt{6ab} = \frac{a}{b}\sqrt{6ab}.$$

$$7. \quad 2\left(\frac{5a^5b}{8c^3}\right)^{\frac{1}{4}} = 2\left(\frac{5a^5b}{8c^3} \times \frac{2c}{2c}\right)^{\frac{1}{4}} = 2\left(\frac{10a^5bc}{16c^4}\right)^{\frac{1}{4}} \\ = 2\left(\frac{a^4}{16c^4} \times 10abc\right)^{\frac{1}{4}} = 2 \times \frac{a}{2c}(10abc)^{\frac{1}{4}} \\ = \frac{a}{c}(10abc)^{\frac{1}{4}}.$$

(ART. 235, pp. 203, 204.)

$$2. \quad 3ax = (3ax)^{\frac{2}{3}} = \sqrt[3]{(3ax)^2} = \sqrt[3]{9a^2x^2}.$$

$$3. \quad -5a^2b = \sqrt[3]{(-5a^2b)^3} = \sqrt[3]{-125a^6b^3}.$$

$$4. \quad 2x-3 = ((2x-3)^2)^{\frac{1}{2}} = (4x^2-12x+9)^{\frac{1}{2}}.$$

$$5. \quad 2a^2x^3y = \sqrt[4]{(2a^2x^3y)^4} = \sqrt[4]{16a^8x^{12}y^4}, \text{ Ans.}$$

$$6. \quad \frac{3ax^3}{2by^4} = \sqrt[5]{\frac{(3ax^3)^5}{(2by^4)^5}} = \sqrt[5]{\frac{243a^5x^{15}}{32b^5y^{20}}}.$$

(ART. 236, pp. 204, 205.)

$$2. \quad 5\sqrt[3]{xy-1} = \sqrt[3]{5^3(xy-1)} = \sqrt[3]{125xy-125}.$$

$$3. \quad 2a^3\sqrt[4]{3ab} = a^3\sqrt[4]{2^4 \times 3ab} = a^2\sqrt[4]{48ab}.$$

$$4. \quad (a+b)\sqrt{c} = \sqrt{c(a+b)^2} = \sqrt{c(a^2+2ab+b^2)} \\ = (a^2c+2abc+b^2c)^{\frac{1}{2}}.$$

$$5. \quad \frac{a}{b}\sqrt{\frac{b^2c}{a^2-b^2}} = \sqrt{\frac{a^2}{b^2}\left(\frac{b^2c}{a^2-b^2}\right)} = \sqrt{\frac{a^2b^2c}{a^2b^2-b^4}}.$$

(ART. 237, pp. 205, 206.)

$$3. \quad 3\sqrt[3]{3} = 3(3)^{\frac{1}{3}} = 3(3)^{\frac{3}{9}} = 3\sqrt[9]{3^3} = 3\sqrt[9]{27}.$$

$$4. \quad \begin{cases} \sqrt[4]{a} = a^{\frac{1}{4}} = a^{\frac{3}{12}} = (a^3)^{\frac{1}{12}}. \\ (5b)^{\frac{1}{2}} = (5b)^{\frac{2}{4}} = [(5b)^2]^{\frac{1}{4}} = (25b^2)^{\frac{1}{4}}. \\ (a^2+b^2)^{\frac{1}{2}} = (a^2+b^2)^{\frac{4}{8}} = [(a^2+b^2)^4]^{\frac{1}{8}}. \end{cases}$$

$$5. \quad \begin{cases} \sqrt{a} = a^{\frac{1}{2}} = a^{\frac{5}{10}} = \sqrt[10]{a^5}. \\ \sqrt[3]{a-b} = (a-b)^{\frac{1}{3}} = (a-b)^{\frac{10}{30}} = \sqrt[30]{(a-b)^{10}}. \\ \sqrt[5]{a+b} = (a+b)^{\frac{1}{5}} = (a+b)^{\frac{6}{30}} = \sqrt[30]{(a+b)^6}. \end{cases}$$

ADDITION OF RADICALS.

(ART. 238, pp. 206-208.)

$$3. \quad 5\sqrt{98x} = 5\sqrt{49 \times 2x} = 5 \times 7\sqrt{2x} = 35\sqrt{2x}; \\ 35\sqrt{2x} + 10\sqrt{2x} = 45\sqrt{2x}.$$

$$4. \quad \sqrt[3]{48a} = \sqrt[3]{8 \times 6a} = 2\sqrt[3]{6a} \\ \sqrt[3]{162a} = \sqrt[3]{27 \times 6a} = 3\sqrt[3]{6a} \\ \text{Sum} = 5\sqrt[3]{6a}$$

$$5. \quad \sqrt[4]{32} = \sqrt[4]{16 \times 2} = 2\sqrt[4]{2}; \quad 2\sqrt[4]{2} + 5\sqrt[4]{2} = 7\sqrt[4]{2}.$$

$$6. \quad \sqrt{3a^2b} = \sqrt{a^2 \times 3b} = a\sqrt{3b} \\ \sqrt{3x^2b} = \sqrt{x^2 \times 3b} = x\sqrt{3b} \\ \text{Sum} = (a+x)\sqrt{3b}$$

$$7. \quad 5\sqrt{20a^2x} = 5\sqrt{4a^2 \times 5x} = 5 \times 2a\sqrt{5x} = 10a\sqrt{5x} \\ 3\sqrt{45a^2x} = 3\sqrt{9a^2 \times 5x} = 3 \times 3a\sqrt{5x} = 9a\sqrt{5x} \\ \text{Sum} = 19a\sqrt{5x}$$

$$8. \quad (3a^2b)^{\frac{1}{2}} = (a^2 \times 3b)^{\frac{1}{2}} = a(3b)^{\frac{1}{2}} \\ (27a^2b)^{\frac{1}{2}} = (9a^2 \times 3b)^{\frac{1}{2}} = 3a(3b)^{\frac{1}{2}} \\ \text{Sum} = 4a(3b)^{\frac{1}{2}}$$

$$9. \quad (45c^3)^{\frac{1}{2}} = (9c^2 \times 5c)^{\frac{1}{2}} = 3c(5c)^{\frac{1}{2}} \\ (80c^3)^{\frac{1}{2}} = (16c^2 \times 5c)^{\frac{1}{2}} = 4c(5c)^{\frac{1}{2}} \\ (5a^3c)^{\frac{1}{2}} = (a^2 \times 5c)^{\frac{1}{2}} = a(5c)^{\frac{1}{2}} \\ \text{Sum} = (7c+a)(5c)^{\frac{1}{2}}, \text{ Ans.}$$

$$10. \quad \sqrt[3]{b^4y} = \sqrt[3]{b^3 \times by} = b\sqrt[3]{by} \\ \sqrt[3]{b^4y^4} = \sqrt[3]{y^3 \times by} = y\sqrt[3]{by} \\ \text{Sum} = (b+y)\sqrt[3]{by}$$

$$\begin{aligned}
 12. \quad 12\sqrt[3]{\frac{1}{4}} &= 12\sqrt[3]{\frac{2}{8}} = 12\sqrt[3]{\frac{1}{8}} \times 2 = 12 \times \frac{1}{2} \sqrt[3]{2} = 6\sqrt[3]{2} \\
 3\sqrt[3]{\frac{1}{32}} &= 3\sqrt[3]{\frac{2}{64}} = 3\sqrt[3]{\frac{1}{64}} \times 2 = 3 \times \frac{1}{4} \sqrt[3]{2} = \frac{3}{4} \sqrt[3]{2} \\
 \text{Sum} &= \frac{27}{4} \sqrt[3]{2}
 \end{aligned}$$

13. $\sqrt{4x} = 2\sqrt{x}$. The radicals being all dissimilar, their addition can only be *indicated* by the sign +.

$$\begin{aligned}
 14. \quad \sqrt{4x+4} &= \sqrt{4(x+1)} = 2\sqrt{x+1}; \\
 \sqrt{x+1} + 2\sqrt{x+1} &= 3\sqrt{x+1}.
 \end{aligned}$$

15. $(x^3y)^{\frac{1}{3}} = xy^{\frac{1}{3}}$; $(xy^3)^{\frac{1}{3}} = x^{\frac{1}{3}}y$; $(8a^4y)^{\frac{1}{3}} = (8a^3 \times ay)^{\frac{1}{3}} = 2a(ay)^{\frac{1}{3}}$. These radicals are all dissimilar.

$$\begin{aligned}
 16. \quad \frac{1}{2}\sqrt{a^2bc} &= \frac{1}{2} \times a\sqrt{bc} = \frac{a}{2}\sqrt{bc} \\
 \frac{1}{3}\sqrt{4bca^2} &= \frac{1}{3} \times 2x^2\sqrt{bc} = \frac{2x^2}{3}\sqrt{bc} \\
 &= \left(\frac{a}{2} + \frac{2x^2}{3}\right)\sqrt{bc}
 \end{aligned}$$

$$\begin{aligned}
 17. \quad \sqrt[3]{16a^3b} &= \sqrt[3]{8a^3 \times 2b} = 2a\sqrt[3]{2b} \\
 \sqrt[3]{54a^3b} &= \sqrt[3]{27a^3 \times 2b} = 3a\sqrt[3]{2b} \\
 &= 5a\sqrt[3]{2b}
 \end{aligned}$$

$$\sqrt{4a^2b} = 2a\sqrt{b}; \sqrt{a^2b} = a\sqrt{b}; 2a\sqrt{b} + a\sqrt{b} = 3a\sqrt{b}$$

It will be seen that $5a\sqrt[3]{2b}$ and $3a\sqrt{b}$ are dissimilar, because the radicals are of a different degree, while the quantities under the radical sign are also different. Their addition can, therefore, be only *indicated*.

SUBTRACTION OF RADICALS.

(ART. 239, pp. 208, 209.)

$$\begin{aligned}
 2. \quad \sqrt{45a} &= \sqrt{9 \times 5a} = 3\sqrt{5a}; \quad 3\sqrt{5a} - \sqrt{5a} = 2\sqrt{5a} \\
 3. \quad \sqrt[3]{192} &= \sqrt[3]{64 \times 3} = 4\sqrt[3]{3} \\
 \sqrt[3]{24} &= \sqrt[3]{8 \times 3} = 2\sqrt[3]{3} \\
 \text{Difference} &= 2\sqrt[3]{3}
 \end{aligned}$$

$$4. \quad (9a^4x)^{\frac{1}{2}} = 3a^2x^{\frac{1}{2}}; (4a^4x)^{\frac{1}{2}} = 2a^2x^{\frac{1}{2}}; 3a^2x^{\frac{1}{2}} - 2a^2x^{\frac{1}{2}} \\ = a^2x^{\frac{1}{2}} = a^2\sqrt{x}.$$

$$5. \quad b\sqrt[3]{8a^3b} = b\sqrt[3]{8a^3 \times b} = 2a^2b\sqrt[3]{b} \\ a\sqrt[3]{a^3b^4} = a\sqrt[3]{a^3b^3 \times b} = a^2b\sqrt[3]{b} \\ \text{Difference} = \underline{a^2b\sqrt[3]{b}}$$

6. The radicals are already similar; hence we merely take the difference of the coefficients 4 and 3, and annex the common radical.

$$7. \quad \sqrt{108ax^3} = \sqrt{36x^2 \times 3a} = 6x\sqrt{3a} \\ \sqrt{48ax^3} = \sqrt{16x^2 \times 3a} = 4x\sqrt{3a} \\ \text{Difference} = 2x\sqrt{3a} \text{ Ans.}$$

$$8. \quad \sqrt{\frac{3}{4}} = \sqrt{\frac{1}{4} \times 3} = \frac{1}{2}\sqrt{3} = \frac{3}{2}\sqrt{\frac{1}{3}} \\ \sqrt{\frac{1}{3}} = \sqrt{\frac{1}{3} \times \frac{3}{3}} = \sqrt{\frac{3}{9}} = \sqrt{\frac{1}{3} \times 3} = \frac{1}{3}\sqrt{3} = \frac{2}{3}\sqrt{\frac{3}{2}} \\ \text{Difference} = \frac{1}{6}\sqrt{3}$$

$$9. \quad 2\sqrt{3a^2b^2c} = 2\sqrt{a^2b^2 \times 3c} = 2ab\sqrt{3c} \\ \sqrt{5ab^3} = \sqrt{b^2 \times 5ab} = b\sqrt{5ab} \\ \text{Difference} = \underline{2ab\sqrt{3c} - b\sqrt{5ab}}$$

$$10. \quad \sqrt[4]{32a} = \sqrt[4]{16 \times 2a} = 2\sqrt[4]{2a} \\ 2\sqrt[4]{40a} = 2\sqrt[4]{8 \times 5a} = 4\sqrt[4]{5a} \\ \text{Difference} = 2\sqrt[4]{2a} - 4\sqrt[4]{5a} = 2(\sqrt[4]{2a} - 2\sqrt[4]{5a})$$

$$11. \quad \sqrt[3]{8a^3b + 16a^4} = \sqrt[3]{8a^3(b + 2a)} = 2a\sqrt[3]{2a + b} \\ \sqrt[3]{b^4 + 2ab^3} = \sqrt[3]{b^3(b + 2a)} = b\sqrt[3]{2a + b} \\ \text{Difference} = \underline{(2a - b)\sqrt[3]{2a + b}}$$

MULTIPLICATION OF RADICALS.

(ART. 240, pp. 209-211.)

4. $6\sqrt{54} \times 3\sqrt{2} = 6 \times 3\sqrt{54} \times 2 = 18\sqrt{108}$
 $= 18\sqrt{36 \times 3} = 18 \times 6\sqrt{3} = 108\sqrt{3}$, Ans.
5. $7\sqrt{axy} \times 3\sqrt{2ax} = 21\sqrt{2a^2x^2y} = 21ax\sqrt{2y}$.
6. $a\sqrt[3]{x} \times b\sqrt[3]{y} = a\sqrt[3]{x^3} \times b\sqrt[3]{y^3} = ab\sqrt[3]{x^3y^3}$.
7. $4\sqrt[3]{ax} \times 3\sqrt[3]{xy} = 4\sqrt[3]{a^2x^2} \times 3\sqrt[3]{x^3y^3} = 12\sqrt[3]{a^2x^5y^3}$.
8. $\frac{1}{2}\sqrt{6} \times \frac{1}{5}\sqrt{9} = \frac{1}{2} \times \frac{1}{5}\sqrt{6 \times 9} = \frac{1}{2} \times \frac{1}{5} \times 3\sqrt{6}$
 $= \frac{1}{10}\sqrt{6}$.

NOTE. The second of the given radicals, $\frac{1}{5}\sqrt{9}$, is rational, and reduces to $\frac{1}{5} \times 3 = \frac{3}{5}$. Multiplying $\frac{1}{2}\sqrt{6}$ by $\frac{3}{5}$ we obtain $\frac{1}{10}\sqrt{6}$, the same result as before.

9. $2\sqrt[3]{\frac{2}{3}} \times 3\sqrt[3]{\frac{2}{3}} = 2 \times 3\sqrt[3]{\frac{2}{3} \times \frac{2}{3}} = 6\sqrt[3]{\frac{4}{9}} = 6\sqrt[3]{\frac{2}{3} \times \frac{2}{3}}$
 $= 6\sqrt[3]{\frac{2^2}{3^2}} = 6\sqrt[3]{\frac{2^2}{3^2} \times 15} = \frac{6}{5}\sqrt[3]{15} = 2\sqrt[3]{15}$.
10. $3b^{\frac{1}{2}} \times 4a^{\frac{1}{2}} = 12a^{\frac{1}{2}}b^{\frac{1}{2}} = 12a^{\frac{3}{12}}b^{\frac{4}{12}} = 12\sqrt[12]{a^3b^4}$.
11. $3a\sqrt[3]{8a^2} \times 2b\sqrt[3]{4a^2c} = 3a \times 2b\sqrt[3]{8a^2 \times 4a^2c}$
 $= 6ab\sqrt[3]{32a^4c} = 6ab\sqrt[3]{16a^4 \times 2c} = 6ab \times 2a\sqrt[3]{2c}$
 $= 12a^2b\sqrt[3]{2c}$.
12. $(a+b)^{\frac{1}{2}} \times (a+b)^{\frac{2}{3}} = (a+b)^{\frac{3}{6}} \times (a+b)^{\frac{4}{6}} = (a+b)^{\frac{7}{6}}$.
13. $\sqrt{\frac{2ax}{3c}} \times \sqrt{\frac{9ab}{2x}} = \sqrt{\frac{2ax}{3c} \times \frac{9ab}{2x}} = \sqrt{\frac{18a^2b}{6c}}$
 $= \sqrt{\frac{3a^2b}{c}}$.
14. $\sqrt[3]{6a^3bc^{-1}} \times \sqrt[3]{3^{-1}a^{-1}bc^3} = \sqrt[3]{6a^3bc^{-1} \times 3^{-1}a^{-1}bc^3}$
 $= \sqrt[3]{\frac{6}{3}a^2b^2c^2} = \sqrt[3]{2b^2c^2}$.

(ART. 241, pp. 211, 212.)

$$(2.)$$

$$\frac{4 + 2\sqrt{2}}{2 - \sqrt{2}}$$

$$\frac{8 + 4\sqrt{2}}{-4\sqrt{2} - 4}$$

$$\frac{8}{-4} = 4$$

$$3. (a+x)^{\frac{2}{3}} \times (a-x)^{\frac{2}{3}} = [(a+x)(a-x)]^{\frac{2}{3}} = (a^2 - x^2)^{\frac{2}{3}}.$$

$$4. \sqrt{a+b} \times \sqrt{a+b} = \sqrt{(a+b)^2} = a+b.$$

NOTE. It will be seen that multiplying the square root of any quantity by itself removes the radical.

$$(5.)$$

$$\frac{\frac{1}{2} + \frac{1}{2}\sqrt{5}}{\frac{3}{4} + \frac{1}{4}\sqrt{5}}$$

$$\frac{\frac{3}{8} + \frac{3}{8}\sqrt{5}}{\frac{1}{8}\sqrt{5} + \frac{5}{8}}$$

$$\frac{3}{8} + \frac{3}{8}\sqrt{5} + \frac{5}{8} = 1 + \frac{1}{2}\sqrt{5}$$

$$(6.)$$

$$\frac{\sqrt{x} + \sqrt{a+x}}{\sqrt{a+x}}$$

$$\sqrt{ax + x^2} + a + x$$

$$(7.)$$

$$\frac{a^{\frac{3}{2}} + b^{\frac{3}{2}}}{a^{\frac{1}{2}} - 2b^{\frac{3}{2}}}$$

$$\frac{a^{\frac{4}{2}} + a^{\frac{1}{2}}b^{\frac{3}{2}}}{-2a^{\frac{3}{2}}b^{\frac{3}{2}} - 2b^{\frac{13}{6}}}$$

$$a^2 + a^{\frac{1}{2}}b^{\frac{3}{2}} - 2a^{\frac{3}{2}}b^{\frac{3}{2}} - 2b^{\frac{13}{6}}$$

DIVISION OF RADICALS.

(ART. 242, pp. 213-215.)

$$3. \frac{\sqrt{40}}{\sqrt{2}} = \sqrt{\frac{40}{2}} = \sqrt{20} = \sqrt{4 \times 5} = 2\sqrt{5}.$$

$$4. \frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} = \left(\frac{a}{b}\right)^{\frac{1}{n}} = \sqrt[n]{\frac{a}{b}}.$$

$$5. \sqrt[3]{\frac{135}{5}} = \sqrt[3]{\frac{135}{5}} = \sqrt[3]{27} = 3. \quad \text{Ans. 3.}$$

NOTE. The same result can be obtained by reducing $\sqrt[3]{135}$ to its simplest form, $3\sqrt[3]{5}$, and then dividing by $\sqrt[3]{5}$.

$$6. \frac{4\sqrt[4]{a^2}}{2\sqrt[4]{a}} = \frac{4\sqrt[4]{a^2}}{2\sqrt[4]{a}} = \frac{4}{2}\sqrt[4]{\frac{a^2}{a}} = 2\sqrt[4]{a^2} = 2a\sqrt[4]{a}.$$

$$7. \frac{4\sqrt[4]{ax}}{3\sqrt[4]{xy}} = \frac{4\sqrt[4]{a^2x^2}}{3\sqrt[4]{x^2y^2}} = \frac{4}{3}\sqrt[4]{\frac{a^2x^2}{x^2y^2}} = \frac{4}{3}\sqrt[4]{\frac{a^2}{y^2}}.$$

$$8. \frac{bc\sqrt[4]{ab}}{b\sqrt[4]{a}} = \frac{bc}{b}\sqrt[4]{\frac{ab}{a}} = c\sqrt[4]{b}.$$

$$9. \frac{(1-x^2)^{\frac{1}{2}}}{(1+x)^{\frac{1}{2}}} = \left(\frac{1-x^2}{1+x}\right)^{\frac{1}{2}} = (1-x)^{\frac{1}{2}}.$$

$$10. \sqrt{\frac{a}{b}} \div \sqrt{\frac{c}{d}} = \sqrt{\frac{a}{b} \div \frac{c}{d}} = \sqrt{\frac{a}{b} \times \frac{d}{c}} = \sqrt{\frac{ad}{bc}}.$$

$$\begin{aligned} 11. \frac{1}{2}\sqrt[3]{\frac{1}{2}} \div \frac{1}{3}\sqrt[3]{\frac{1}{3}} &= \left(\frac{1}{2} \div \frac{1}{3}\right)\sqrt[3]{\frac{1}{2} \div \frac{1}{3}} = \left(\frac{1}{2} \times \frac{3}{1}\right)\sqrt[3]{\frac{1}{2} \times \frac{3}{1}} \\ &= \frac{3}{2}\sqrt[3]{\frac{3}{2}} = \frac{3}{2}\sqrt[3]{\frac{3}{2} \times \frac{2}{1}} = \frac{3}{2}\sqrt[3]{12} = \frac{3}{2} \times \frac{1}{2}\sqrt[3]{12} \\ &= \frac{3}{4}\sqrt[3]{12}. \end{aligned}$$

$$\begin{aligned} 12. \frac{\sqrt{72} + \sqrt{32-4}}{\sqrt{8}} &= \sqrt{\frac{72}{8}} + \sqrt{\frac{28}{8}} = \sqrt{9} + \sqrt{\frac{7}{2}} \\ &= 3 + \sqrt{\frac{7}{2} \times \frac{2}{2}} = 3 + \sqrt{\frac{14}{4}} = 3 + \frac{1}{2}\sqrt{14}. \end{aligned}$$

13. The radicals, being alike, cancel each other in dividing, and there is nothing left but the coefficients m and n , which give $\frac{m}{n}$ as the result.

$$\begin{aligned} 14. \frac{\sqrt{a^2-b^2}}{a-b} &= \frac{\sqrt{(a+b)(a-b)}}{\sqrt{(a-b)^2}} = \sqrt{\frac{(a+b)(a-b)}{(a-b)^2}} \\ &= \sqrt{\frac{a+b}{a-b}}. \end{aligned}$$

(15.)

$$\begin{array}{r}
 a - 2b^{\frac{1}{2}} \bigg| a^2 + a b^{\frac{1}{2}} - 6b(a + 3b^{\frac{1}{2}}) \\
 \underline{a^2 - 2ab^{\frac{1}{2}}} \\
 3ab^{\frac{1}{2}} - 6b \\
 \underline{3ab^{\frac{1}{2}} - 6b} \\
 0
 \end{array}$$

(16.)

$$\begin{array}{r}
 a^{\frac{1}{2}} - 4b^{\frac{1}{2}} \bigg| a^2 + 2a^{\frac{1}{2}}b^{\frac{3}{2}} - 4a^{\frac{3}{2}}b^{\frac{1}{2}} - 8b^{\frac{7}{2}}(a^{\frac{3}{2}} + 2b^{\frac{3}{2}}) \\
 \underline{a^2 - 4a^{\frac{3}{2}}b^{\frac{1}{2}}} \\
 2a^{\frac{1}{2}}b^{\frac{3}{2}} - 8b^{\frac{7}{2}} \\
 \underline{2a^{\frac{1}{2}}b^{\frac{3}{2}} - 8b^{\frac{7}{2}}} \\
 0
 \end{array}$$

INVOLUTION OF RADICALS.

(ART. 243, pp. 215, 216.)

3. $(5a^{\frac{1}{3}})^2 = 5^2 a^{\frac{2}{3}} = 25a^{\frac{2}{3}}.$
4. $(5a\sqrt[3]{y})^3 = 5^3 a^3 \sqrt[3]{y^3} = 125a^3 y.$
5. $(x^2\sqrt[3]{6})^3 = (x^2)^3 \sqrt[3]{6^3} = x^6 \sqrt[3]{36}.$
6. $(4a^2\sqrt{3c})^4 = 4^4 a^8 \sqrt{3^4 c^4} = 4^4 a^8 \times 3^2 c^2 = 2304 a^8 c^2, \text{Ans.}$
7. $(3\sqrt[3]{25ax})^3 = 3^3 \sqrt[3]{25^3 ax^3} = 9 \times 5\sqrt{ax} = 45\sqrt{ax}.$

8.

$$\begin{array}{r}
 \sqrt{3} - \sqrt{2} \\
 \sqrt{3} - \sqrt{2} \\
 \hline
 3 - \sqrt{6} \\
 - \sqrt{6} + 2 \\
 \hline
 3 - 2\sqrt{6} + 2 = 5 - 2\sqrt{6} \\
 5 - 2\sqrt{6} \\
 5 - 2\sqrt{6} \\
 \hline
 25 - 10\sqrt{6} \\
 - 10\sqrt{6} + 24 \\
 \hline
 25 - 20\sqrt{6} + 24 = 49 - 20\sqrt{6}
 \end{array}$$

9. $(\sqrt[3]{2a})^3 = \sqrt[3]{2^3 a^3}$.
10. $(\sqrt{3} + x\sqrt{3})^2 = \sqrt{3^2} + 2x\sqrt{3^2} + x^2\sqrt{3^2}$
 $= 3 + 6x + 3x^2$.
11. $(x^{\frac{1}{2}} - y^{-\frac{2}{3}})^2 = (x^{\frac{1}{2}})^2 - 2x^{\frac{1}{2}}y^{-\frac{2}{3}} + (y^{-\frac{2}{3}})^2$
 $= x - 2x^{\frac{1}{2}}y^{-\frac{2}{3}} + y^{-\frac{4}{3}}$.

EVOLUTION OF RADICALS.

(ART. 244, pp. 216-218.)

3. $(\sqrt{ab})^{\frac{1}{2}} = \sqrt[4]{ab}$.
4. $\sqrt{4}\sqrt[3]{5c} = \sqrt{4} \times \sqrt[3]{5c} = 2\sqrt[3]{5c}$.
5. $\sqrt[3]{a^{-1}(xy)^{\frac{1}{2}}} = a^{-\frac{1}{3}}(xy)^{\frac{1}{6}} = a^{-\frac{1}{3}}x^{\frac{1}{6}}y^{\frac{1}{6}}$.
6. $\sqrt{25}\sqrt[3]{4a^2b} = \sqrt{25} \times \sqrt[3]{4a^2b} = 5\sqrt[3]{4a^2b}$, Ans.
7. $a\sqrt{a} = \sqrt{a^2} \times a = \sqrt{a^3}$; $(\sqrt{a^3})^{\frac{1}{3}} = \sqrt[3]{a^3} = \sqrt[3]{a}$.
8. $\frac{a}{3}\sqrt[3]{\frac{a}{3}} = \sqrt{\frac{a^2}{3^2} \times \frac{a}{3}} = \sqrt{\frac{a^3}{3^3}}$; $(\sqrt[3]{\frac{a^3}{3^3}})^{\frac{1}{3}} = \sqrt[3]{\frac{a}{3}} = \sqrt{\frac{a}{3} \times \frac{3}{3}}$
 $= \sqrt{\frac{3a}{9}} = \frac{1}{3}\sqrt{3a}$.
9. $\sqrt[3]{125x^{\frac{1}{2}}} = 5x^{\frac{1}{6}}$.
10. $\sqrt[4]{64a^3b^4}\sqrt{2cd} = \sqrt[4]{16a^3b^4}\sqrt[4]{4^2} \times \sqrt{2cd} = 2a^{\frac{3}{4}}b\sqrt{32cd}$.

(11.)

$$\begin{array}{r|l}
 x^{\frac{4}{3}} + 6x^{\frac{2}{3}}y^{\frac{1}{3}} + 9y & x^{\frac{2}{3}} + 3y^{\frac{1}{3}} \\
 x^{\frac{4}{3}} & \\
 \hline
 2x^{\frac{2}{3}} + 3y^{\frac{1}{3}} & 6x^{\frac{2}{3}}y^{\frac{1}{3}} + 9y \\
 & 6x^{\frac{2}{3}}y^{\frac{1}{3}} + 9y
 \end{array}$$

RATIONALIZATION.

(ART. 246, pp. 218, 219.)

3. The factor $x^{\frac{1}{4}}$ will rationalize $x^{\frac{3}{4}}$, because $x^{\frac{3}{4}} \times x^{\frac{1}{4}} = x^1 = x$.

$$4. \quad 4 \sqrt[3]{a b^2} = 4 a^{\frac{1}{3}} b^{\frac{2}{3}}; \quad 4 a^{\frac{1}{3}} b^{\frac{2}{3}} \times a^{\frac{2}{3}} b^{\frac{1}{3}} = 4 a b; \\ a^{\frac{2}{3}} b^{\frac{1}{3}} = \sqrt[3]{a^2 b}.$$

NOTE. It is evident, without the use of fractional exponents, that $4 \sqrt[3]{a b^2} \times \sqrt[3]{a^2 b} = 4 \sqrt[3]{a^3 b^3} = 4 a b$. In general, such a radical quantity is rationalized by means of another radical of the same degree, with such exponents as will make the product a perfect power of the same degree as the radical, and thus remove the radical sign.

5. $a^{-\frac{1}{2}} \times a^{\frac{3}{2}} = a^1 = a$. The factor $a^{-\frac{1}{2}}$ will rationalize $a^{\frac{3}{2}}$, as the product will be $a^{-1} = \frac{1}{a}$; but it will not make the exponent positive, that is, it will not make the product integral.

NOTE. The rule, as stated in the Algebra, is intended to apply to radicals in their simplest form. (Art. 233.) Thus, to rationalize $\sqrt[3]{a^2}$, we first reduce it to $a \sqrt[3]{a}$, and then multiply by such a factor as will rationalize $\sqrt[3]{a}$, without reference to the coefficient a , that being already rational. If, however, we are required to rationalize $\sqrt[3]{a^2}$, or $a^{\frac{2}{3}}$, as it stands, we must multiply the given surd by the same quantity with such a fractional exponent as, when added to the fractional exponent of the given quantity, shall be equal to an integer. Thus, $a^{\frac{2}{3}} \times a^{\frac{1}{3}} = a^1 = a$. If the radical signs are to be used throughout, we must multiply by such a radical, of the same degree, as will make the quantity under the radical a perfect power of the same degree as the radical. Thus, $\sqrt[3]{a^2} \times \sqrt[3]{a} = \sqrt[3]{a^3} = a$.

(ART. 247, pp. 219, 220.)

$$2. \quad (a + \sqrt{b})(a - \sqrt{b}) = a^2 - b.$$

$$3. \quad (\sqrt{5} - \sqrt{1})(\sqrt{5} + \sqrt{1}) = 5 - 1 = 4.$$

NOTE. It will be observed that the radical sign over 1 does not change its value, and the multiplication may be thus expressed :

$$(\sqrt{5} - 1)(\sqrt{5} + 1) = 5 - 1 = 4.$$

(ART. 249, pp. 220, 221.)

$$2. \frac{\sqrt{a}}{\sqrt{b}} = \frac{\sqrt{a}}{\sqrt{b}} \times \frac{\sqrt{a}}{\sqrt{a}} = \frac{a}{\sqrt{ab}}.$$

$$3. \frac{1}{\sqrt{3}+1} = \frac{1}{\sqrt{3}+1} \times \frac{\sqrt{3}-1}{\sqrt{3}-1} = \frac{\sqrt{3}-1}{3-1} = \frac{\sqrt{3}-1}{2}.$$

$$4. \frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3}, \text{ Ans.}$$

$$5. \frac{a}{\sqrt{b}+\sqrt{c}} = \frac{a}{\sqrt{b}+\sqrt{c}} \times \frac{\sqrt{b}-\sqrt{c}}{\sqrt{b}-\sqrt{c}} = \frac{a(\sqrt{b}-\sqrt{c})}{b-c}.$$

$$6. \frac{\sqrt{2}}{3-\sqrt{2}} = \frac{\sqrt{2}}{3-\sqrt{2}} \times \frac{3+\sqrt{2}}{3+\sqrt{2}} = \frac{3\sqrt{2}+2}{9-2} = \frac{3\sqrt{2}+2}{7}.$$

$$7. \frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}} = \frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}} \times \frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}+\sqrt{y}} = \frac{(\sqrt{x}+\sqrt{y})^2}{x-y}.$$

IMAGINARY QUANTITIES.

(ART. 255, pp. 223, 224.)

1. The multiplication may be effected as indicated in the Algebra, or as follows:

$$\sqrt{-9} \times \sqrt{-4} = 3\sqrt{-1} \times 2\sqrt{-1} = 6(-1) = -6.$$

$$3. (1+\sqrt{-1})(1-\sqrt{-1}) = 1-(-1) = 1+1 = 2.$$

$$4. \frac{a\sqrt{-1}}{b\sqrt{-1}} = \frac{a}{b}, \text{ as the radicals cancel each other.}$$

$$5. \frac{6\sqrt{-3}}{2\sqrt{-4}} = \frac{6\sqrt{-3}}{4\sqrt{-1}} = \frac{6}{4} \sqrt{\frac{-3}{-1}} = \frac{3}{2} \sqrt{3}.$$

$$6. \frac{2\sqrt{-10}}{-5\sqrt{-2}} = -\frac{2}{5} \sqrt{\frac{-10}{-2}} = -\frac{2}{5} \sqrt{5}.$$

RADICAL EQUATIONS

LEADING TO SIMPLE EQUATIONS.

(ART. 257, pp. 224-226.)

4. Given $\sqrt{x+1} - 2 = 3$
 Transposing and uniting, $\sqrt{x+1} = 5$
 Squaring, $x+1 = 25$
 Transposing and uniting, $x = 24$
5. Given $\sqrt{x+7} = \sqrt{x} + 1$
 Squaring, $x+7 = x+2\sqrt{x}+1$
 Transposing and uniting, $-2\sqrt{x} = -6$
 Dividing by -2 , $\sqrt{x} = 3$
 Squaring, $x = 9$
6. Given $2 + \sqrt{3x-30} = 5$
 Transposing and uniting, $\sqrt{3x-30} = 3$
 Squaring, $3x-30 = 9$
 Transposing and uniting, $3x = 39$
 Whence, $x = 13$
7. Given $\sqrt{3-x} + 6 = 8 - 1$
 Transposing and uniting, $\sqrt{3-x} = 1$
 Squaring, $3-x = 1$
 Whence, $x = 2$, Ans.
8. Given $x^{\frac{1}{2}} - 7 = -3$
 Transposing and uniting, $x^{\frac{1}{2}} = 4$
 Squaring, $x = 16$
9. Given $\sqrt{x+6} = 3$
 Squaring, $x+6 = 9$
 Whence, $x = 3$
10. Given $\sqrt{4+x} = 4 - \sqrt{x}$
 Squaring, $4+x = 16 - 8\sqrt{x} + x$
 Transposing and uniting, $8\sqrt{x} = 12$

- Dividing by 8, $\sqrt{x} = \frac{3}{2}$
 Squaring, $x = \frac{9}{4} = 2\frac{1}{4}$
11. Given $y^{\frac{1}{3}} + 5 = 11$
 Transposing and uniting, $y^{\frac{1}{3}} = 6$
 Cubing, $y = 216$
12. Given $\sqrt[4]{20 - \sqrt{2}x} - 2 = 0$
 Transposing, $\sqrt[4]{20 - \sqrt{2}x} = 2$
 Involving to the fourth power, $20 - \sqrt{2}x = 16$
 Transposing and uniting, $-\sqrt{2}x = -4$
 Squaring, $2x = 16$
 Whence, $x = 8$
13. Given $\frac{x - ax}{\sqrt{x}} = \frac{\sqrt{x}}{x}$
 Clearing of fractions, $x^2 - ax^2 = x$
 Dividing by x , $x - ax = \frac{1}{x}$
 Whence, $x = \frac{1}{1-a}$

NOTE. Multiplying the given equation by $\sqrt{x}$ will also produce $x - ax = 1$.

14. Given $x(x^{\frac{1}{2}} + x^{-\frac{1}{2}}) = ax^{\frac{1}{2}}$
 Multiplying by $x^{\frac{1}{2}}$, $x(x + 1) = ax$
 Dividing by x , $x + 1 = a$
 Transposing, $x = a - 1$

NOTE. Performing the multiplication indicated in the original equation gives $x^{\frac{3}{2}} + x^{\frac{1}{2}} = ax^{\frac{1}{2}}$, and this divided by $x^{\frac{1}{2}}$ also gives $x + 1 = a$.

15. Given $\frac{\sqrt{x} + 3}{3} + 3 = \frac{3\sqrt{x}}{\sqrt{x} - 3}$
 Clear. of fractions, $x - 9 + 9\sqrt{x} - 27 = 9\sqrt{x}$
 Transposing and uniting, $x = 36$
16. Given $\frac{ax - 1}{\sqrt{ax} + 1} = 4 + \frac{\sqrt{ax} - 1}{2}$

Perform. the divis. ind., $\sqrt{ax} - 1 = 4 + \frac{\sqrt{ax} - 1}{2}$

Clearing of fractions, $2\sqrt{ax} - 2 = 8 + \sqrt{ax} - 1$

Transposing and uniting, $\sqrt{ax} = 9$

Squaring, $ax = 81$

Whence, $x = \frac{81}{a}$

QUADRATIC EQUATIONS.

PURE QUADRATIC EQUATIONS.

(ART. 263, pp. 227-231.)

3. Given $3x^2 - 2 = 2x^2 + 2$

Transposing and uniting, $x^2 = 4$

Evolving, $x = \pm 2$

4. Given $\frac{5x^2}{2} + x^2 = 126$

Clearing of fractions, $5x^2 + 2x^2 = 252$

Uniting terms, $7x^2 = 252$

Dividing by 7, $x^2 = 36$

Evolving, $x = \pm 6$

5. Given $y^2 = 9a^2$

Evolving, $y = \pm 3a$, Ans.

6. Given $7(2x^2 - 6) + 5(3 - x^2) = 198$

Expanding, $14x^2 - 42 + 15 - 5x^2 = 198$

Transposing and uniting, $9x^2 = 225$

Dividing by 9, $x^2 = 25$

Evolving, $x = \pm 5$

7. Given $(x + 1)^2 = 2x + 17$

Expanding, $x^2 + 2x + 1 = 2x + 17$

Transp. and uniting, $x^2 = 16$

Evolving, $x = \pm 4$

8. Given $x^2 + ab = 5x^2$
 Transp. and uniting, $-4x^2 = -ab$
 Dividing by -4 , $x^2 = \frac{ab}{4}$
 Evolving, $x = \pm \frac{\sqrt{ab}}{2} = \pm \frac{1}{2} \sqrt{ab}$
9. Given $\frac{2x^2 + 10}{15} = 7 - \frac{50 + x^2}{25}$
 Multiplying by 75, $10x^2 + 50 = 525 - 150 - 3x^2$
 Transp. and uniting, $13x^2 = 325$
 Dividing by 13, $x^2 = 25$
 Evolving, $x = \pm 5$
10. Given $(x + 2)^2 = 4x + 5$
 Expanding, $x^2 + 4x + 4 = 4x + 5$
 Transp. and uniting, $x^2 = 1$
 Evolving, $x = \pm 1$
11. Given $(2x - 5)^2 = x^2 - 20x + 73$
 Expanding, $4x^2 - 20x + 25 = x^2 - 20x + 73$
 Transp. and uniting, $3x^2 = 48$
 Dividing by 3, $x^2 = 16$
 Evolving, $x = \pm 4$, Ans.
12. Given $4x - 150x^{-1} = x - 3x^{-1}$
 Multiplying by x , $4x^2 - 150 = x^2 - 3$
 Transp. and uniting, $3x^2 = 147$
 Dividing by 3, $x^2 = 49$
 Evolving, $x = \pm 7$
13. Given $\frac{3}{1+x} + \frac{3}{1-x} = 8$
 Clear. of fractions, $3 - 3x + 3 + 3x = 8 - 8x^2$
 Transp. and uniting, $8x^2 = 2$
 Dividing by 8, $x^2 = \frac{1}{4}$
 Evolving, $x = \pm \frac{1}{2}$

14. Given $x^2 - \frac{a^2}{4} + 3ab = 2a^2 + b^2$
 Clear. of frac., $4x^2 - a^2 + 12ab = 8a^2 + 4b^2$
 Transp. and uniting, $4x^2 = 9a^2 - 12ab + 4b^2$
 Evolving,
 $2x = \pm (3a - 2b)$
 Whence,
 $x = \pm \frac{3a - 2b}{2}$
 Or,
 $x = \pm \left(\frac{3}{2}a - b \right)$
15. Given $c(x^2 + 4ab + 4bc) = a(a + 2c)^2 + dx^2 - a^2b$
 Expanding first member,
 $cx^2 + 4abc + 4bc^2 = a(a + 2c)^2 + dx^2 - a^2b$
 Transposing terms,
 $cx^2 - dx^2 = a(a + 2c)^2 - a^2b - 4abc - 4bc^2$
 Factoring, $(c - d)x^2 = a(a + 2c)^2 - b(a + 2c)^2$
 Or, $(c - d)x^2 = (a + 2c)^2(a - b)$
 Dividing by $(c - d)$, $x^2 = \frac{(a + 2c)^2(a - b)}{c - d}$
 Evolving,
 $x = \pm (a + 2c) \sqrt{\frac{a - b}{c - d}}$
16. Given $\frac{x + 2}{x - 2} + \frac{x - 2}{x + 2} = \frac{13}{6}$
 Clear. of frac., $6(x + 2)^2 + 6(x - 2)^2 = 13(x^2 - 4)$
 Expanding,
 $6x^2 + 24x + 24 + 6x^2 - 24x + 24 = 13x^2 - 52$
 Uniting terms,
 $12x^2 + 48 = 13x^2 - 52$
 Transp. and uniting,
 $-x^2 = -100$
 Changing signs,
 $x^2 = 100$
 Evolving,
 $x = \pm 10$
17. Given $x - \frac{3x^2 - 2}{5x} = 3x^{-1} - \frac{2x^2 - 5}{3x}$
 Mult. by $15x$, $15x^2 - 9x^2 + 6 = 45 - 10x^2 + 25$
 Uniting terms,
 $6x^2 + 6 = 70 - 10x^2$
 Transp. and uniting,
 $16x^2 = 64$
 Dividing by 16,
 $x^2 = 4$
 Evolving,
 $x = \pm 2$

18. Given $y^2 - \frac{2}{3y^2 + 1} = \frac{10}{9(3y^2 + 1)}$
 Clearing of fractions, $9(3 + y^2) - 18 = 10$
 Expanding, $27 + 9y^2 - 18 = 10$
 Transp. and uniting, $9y^2 = 1$
 Dividing by 9, $y^2 = \frac{1}{9}$
 Evolving, $y = \pm \frac{1}{3}$
 Whence, $y = \pm 3$

19. Given $\frac{14z^2 + 16}{21} - \frac{2z^2 + 8}{8z^2 - 11} = \frac{2z^2}{3}$
 Clearing of fractions,
 $112z^4 - 26z^2 - 176 - 42z^2 - 168 = 112z^4 - 154z^2$
 Transp. and uniting, $86z^2 = 344$
 Dividing by 86, $z^2 = 4$
 Evolving, $z = \pm 2$

20. Given $(a + x)^2 + (a - x)^2 = 2b^2$
 Expanding,
 $a^2 + 3a^2x + 3ax^2 + x^2 + a^2 - 3a^2x + 3ax^2 - x^2 = 2b^2$
 Uniting, $2a^2 + 6ax^2 = 2b^2$
 Dividing by 2, $a^2 + 3ax^2 = b^2$
 Transposing, $3ax^2 = b^2 - a^2$
 Dividing by $3a$, $x^2 = \frac{b^2 - a^2}{3a}$
 Evolving, $x = \pm \sqrt{\frac{b^2 - a^2}{3a}}$

(ART. 264, pp. 231, 232.)

1. Given $22\frac{1}{2} + \sqrt{5(4x^2 - 1)} = 25$
 Transp. and uniting, $\sqrt{5(4x^2 - 1)} = \frac{5}{2}$
 Squaring, $5(4x^2 - 1) = \frac{25}{4}$
 Dividing by 5, $4x^2 - 1 = \frac{5}{4}$
 Transposing and uniting, $4x^2 = \frac{9}{4}$
 Dividing by 4, $x^2 = \frac{9}{16}$
 Evolving, $x = \pm \frac{3}{4}$

2. Given $\sqrt{x^3 + \sqrt{x^4 - a^4}} = a$
 Squaring, $x^3 + \sqrt{x^4 - a^4} = a^2$
 Transposing, $\sqrt{x^4 - a^4} = a^2 - x^3$
 Squaring, $x^4 - a^4 = a^4 - 2a^2x^3 + x^4$
 Transp. and uniting, $2a^2x^3 = 2a^4$
 Dividing by $2a^2$, $x^3 = a^2$
 Evolving, $x = \pm a$

3. Given $\sqrt{x^2 - 16} = \frac{3x}{5}$
 Squaring, $x^2 - 16 = \frac{9x^2}{25}$
 Clear. of fract., $25x^2 - 16 \times 25 = 9x^2$
 Transposing and uniting, $16x^2 = 16 \times 25$
 Dividing by 16, $x^2 = 25$
 Evolving, $x = \pm 5$, Ans.

4. Given $\sqrt{a^2 + x^2} = \sqrt[4]{b^4 + x^4}$
 Inv. to the fourth power, $a^4 + 2a^2x^2 + x^4 = b^4 + x^4$
 Transposing and canceling, $2a^2x^2 = b^4 - a^4$
 Dividing by $2a^2$, $x^2 = \frac{b^4 - a^4}{2a^2}$
 Evolving, $x = \pm \sqrt{\frac{b^4 - a^4}{2a^2}}$

5. Given $\sqrt{\frac{4x^3 + 128}{8x}} = 2\sqrt{x}$
 Squaring, $\frac{4x^3 + 128}{8x} = 4x$
 Clearing of fractions, $4x^3 + 128 = 12x^2$
 Whence, $8x^3 = 128$
 Dividing by 8, $x^3 = 16$
 Evolving, $x = \pm 4$

6. Given $\sqrt{x+a} = \sqrt{x + \sqrt{b^2 + x^2}}$
 Squaring, $x + a = x + \sqrt{b^2 + x^2}$
 Canceling x , $a = \sqrt{b^2 + x^2}$

$$\begin{array}{ll}\text{Squaring,} & a^2 = b^2 + x^2 \\ \text{Whence,} & x^2 = a^2 - b^2 \\ \text{Evolving,} & x = \pm \sqrt{a^2 - b^2}\end{array}$$

$$\begin{array}{ll}7. \text{ Given} & x(10 + x^2)^{\frac{1}{2}} = 5 - x^2 \\ \text{Squaring,} & 10x^2 + x^4 = 25 - 10x^2 + x^4 \\ \text{Transposing and uniting,} & 20x^2 = 25 \\ \text{Dividing by 20,} & x^2 = \frac{5}{4} \\ \text{Evolving,} & x = \pm \sqrt{\frac{5}{4}} = \pm \frac{1}{2}\sqrt{5}\end{array}$$

$$\begin{array}{ll}8. \text{ Given} & x + (a^2 + x^2)^{\frac{1}{2}} = 2a^2(a^2 + x^2)^{-\frac{1}{2}} \\ \text{Mult. by } (a^2 + x^2)^{\frac{1}{2}}, & x(a^2 + x^2)^{\frac{1}{2}} + a^2 + x^2 = 2a^2 \\ \text{Transp. and uniting,} & x(a^2 + x^2)^{\frac{1}{2}} = a^2 - x^2 \\ \text{Squaring,} & a^2x^2 + x^4 = a^4 - 2a^2x^2 + x^4 \\ \text{Transposing and uniting,} & 3a^2x^2 = a^4 \\ \text{Dividing by } 3a^2, & x^2 = \frac{a^2}{3} \\ \text{Evolving,} & x = \pm \frac{a}{\sqrt{3}}\end{array}$$

$$\begin{array}{ll}9. \text{ Given} & \frac{a - \sqrt{a^2 - x^2}}{a + \sqrt{a^2 - x^2}} = b \\ \text{Rationaliz. denom.,} & \frac{(a - \sqrt{a^2 - x^2})^2}{a^2 - (a^2 - x^2)} = b \\ \text{Or,} & \frac{(a - \sqrt{a^2 - x^2})^2}{x^2} = b \\ \text{Evolving,} & \frac{a - \sqrt{a^2 - x^2}}{x} = \pm \sqrt{b} \\ \text{Clear. of fractions,} & a - \sqrt{a^2 - x^2} = \pm x\sqrt{b} \\ \text{Transposing,} & -\sqrt{a^2 - x^2} = -a \pm x\sqrt{b} \\ \text{Squaring,} & a^2 - x^2 = a^2 \mp 2ax\sqrt{b} + bx^2 \\ \text{Transp. and chang. signs,} & x^2 + bx^2 = \pm 2ax\sqrt{b} \\ \text{Dividing by } x, & x + bx = \pm 2a\sqrt{b} \\ \text{Dividing by } 1 + b, & x = \pm \frac{2a\sqrt{b}}{1 + b}\end{array}$$

10. Given

$$\sqrt{\frac{x+3}{x-3}} + \sqrt{\frac{x-3}{x+3}} = 5$$

Squaring,

$$\frac{x+3}{x-3} + 2 + \frac{x-3}{x+3} = 25$$

Transp. and uniting,

$$\frac{x+3}{x-3} + \frac{x-3}{x+3} = 23$$

Clearing of fractions, $(x+3)^2 + (x-3)^2 = 23(x^2-9)$

Or, $x^2 + 6x + 9 + x^2 - 6x + 9 = 23x^2 - 207$

Uniting terms, $2x^2 + 18 = 23x^2 - 207$

Whence, $21x^2 = 225$

Dividing by 21,

$$x^2 = \frac{225}{21} = \frac{75}{7}$$

Evolving,

$$x = \pm \sqrt{\frac{75}{7}} = \pm 5\sqrt{\frac{3}{7}}$$

Or,

$$x = \pm 5\sqrt{\frac{21}{49}} = \pm \frac{5}{7}\sqrt{21}$$

SIMULTANEOUS EQUATIONS.

(ART. 265, pp. 232, 233.)

2. Given

$$\begin{cases} \frac{1}{2}x^2 - 3y^2 = 21 & (1) \\ \frac{1}{2}x + 2y = 0 & (2) \end{cases}$$

Clearing (1) of fractions,

$$x^2 - 9y^2 = 63 \quad (3)$$

Clearing (2) of fractions,

$$x + 4y = 0 \quad (4)$$

From (4),

$$x = -4y \quad (5)$$

Squaring (5),

$$x^2 = 16y^2 \quad (6)$$

Substituting (6) in (3),

$$16y^2 - 9y^2 = 63 \quad (7)$$

Or,

$$7y^2 = 63 \quad (8)$$

Dividing by 7,

$$y^2 = 9 \quad (9)$$

Evolving,

$$y = \pm 3 \quad (10)$$

Substituting in (5), $x = (-4) \times (\pm 3) = \mp 12$

(11)

NOTE. The values of x and y must have different signs; but it matters not which one is $\pm$, and which is $\mp$. (See NOTE, ART. 286, Alg.)

3. Given

$$\begin{cases} 5xy - 3y^2 = 100 & (1) \\ 5x - 4y = 0 & (2) \end{cases}$$

Multiplying (2) by y ,

$$5xy - 4y^2 = 0 \quad (3)$$

$$\begin{array}{ll}
\text{Subtracting (3) from (1),} & y^2 = 100 \quad (4) \\
\text{Evolving,} & y = \pm 10 \quad (5) \\
\text{Substituting in (2),} & 5x \mp 40 = 0 \quad (6) \\
\text{Transposing,} & 5x = \pm 40 \quad (7) \\
\text{Whence,} & x = \pm 8 \quad (8)
\end{array}$$

4. Given

$$\begin{cases} xy + y^2 = 126 & (1) \\ 5y = 2x & (2) \end{cases}$$

$$\text{From (2),} \quad x = \frac{5y}{2} \quad (3)$$

$$\text{Substituting (3) in (1),} \quad \frac{5y^2}{2} + y^2 = 126 \quad (4)$$

$$\text{Clearing of fractions,} \quad 5y^2 + 2y^2 = 252 \quad (5)$$

$$\text{Uniting terms,} \quad 7y^2 = 252 \quad (6)$$

$$\text{Dividing by 7,} \quad y^2 = 36 \quad (7)$$

$$\text{Evolving,} \quad y = \pm 6 \quad (8)$$

$$\text{Substituting in (3),} \quad x = \frac{\pm 30}{2} = \pm 15 \quad \left. \vphantom{\frac{\pm 30}{2}} \right\} \text{Ans. (9)}$$

5. Given

$$\begin{cases} 4x^2 + 7y^2 = 148 & (1) \\ 3x^2 - y^2 = 11 & (2) \end{cases}$$

$$\text{Multiplying (2) by 7,} \quad 21x^2 - 7y^2 = 77 \quad (3)$$

$$\text{Adding (1) and (3),} \quad 25x^2 = 225 \quad (4)$$

$$\text{Dividing by 25,} \quad x^2 = 9 \quad (5)$$

$$\text{Evolving,} \quad x = \pm 3 \quad (6)$$

$$\text{Substituting (5) in (2),} \quad 27 - y^2 = 11 \quad (7)$$

$$\text{Whence,} \quad y^2 = 16 \quad (8)$$

$$\text{Evolving,} \quad y = \pm 4 \quad (9)$$

6. Given

$$\begin{cases} x + y = 3x - 3y & (1) \\ x^2 - y^2 = 56 & (2) \end{cases}$$

$$\text{Transposing in (1),} \quad -2x = -4y \quad (3)$$

$$\text{Dividing by } -2, \quad x = 2y \quad (4)$$

$$\text{Cubing (4),} \quad x^2 = 8y^2 \quad (5)$$

$$\text{Substituting (5) in (2),} \quad 8y^2 - y^2 = 56 \quad (6)$$

$$\text{Uniting terms,} \quad 7y^2 = 56 \quad (7)$$

Dividing by 7, $y^2 = 8$ (8)

Evolving, $y = 2$ (9)

Substituting in (4), $x = 2 \times 2 = 4$ (10)

7. Given $\begin{cases} x^2 + y^2 : x^2 - y^2 :: 17 : 8 \\ x y^2 = 45 \end{cases}$ (1)

From (1), $8x^2 + 8y^2 = 17x^2 - 17y^2$ (2)

From (3), $9x^2 = 25y^2$ (4)

Evolving, $3x = \pm 5y$ (5)

From (5), $x = \pm \frac{5y}{3}$ (6)

Substituting (6) in (2), $\pm \frac{5y^3}{3} = 45$ (7)

Clearing of fractions, $\pm 5y^3 = 135$ (8)

Dividing by ± 5 , $y^3 = \pm 27$ (9)

Evolving, $y = \pm 3$ (10)

Substituting in (6), $x = \pm \frac{5 \times (\pm 3)}{3} = 5$ (11)

NOTE. The value of x can only take the positive sign, for $y^2 = 9$, which, substituted in (2), gives $9x = 45$, and $x = 5$. The same results are obtained if we first eliminate y^2 from (2) and (4), showing that if all the possible roots are taken, the answer is $x = 5$; $y = \pm 3$.

8. Given $\begin{cases} x^4 + y^4 = 97 \\ 9x^2 = 4y^2 \end{cases}$ (1)

From (2), $x^2 = \frac{4y^2}{9}$ (2)

Squaring (3), $x^4 = \frac{16y^4}{81}$ (4)

Substituting (4) in (1), $\frac{16y^4}{81} + y^4 = 97$ (5)

Clearing of fractions, $16y^4 + 81y^4 = 97 \times 81$ (6)

Uniting terms, $97y^4 = 97 \times 81$ (7)

Dividing by 97, $y^4 = 81$ (8)

Evolving, $y = \pm 3$ (9)

Substituting (8) in (1), $x^4 + 81 = 97$ (10)

Transposing and uniting, $x^4 = 16$ (11)

Evolving, $x = \pm 2$ (12)

NOTE. By taking the square root twice, we obtain $y^2 = \pm 9$, $x^2 = \pm 4$, and $y = \pm \sqrt{\pm 9}$, $x = \pm \sqrt{\pm 4}$; that is, $y = \pm 3$, or $\pm 3\sqrt{-1}$, $x = \pm 2$, or $\pm 2\sqrt{-1}$.

PROBLEMS

LEADING TO PURE EQUATIONS.

(ART. 266, pp. 234, 235.)

- | | |
|-----------------|--------------------------------|
| 2. Let | $x =$ the smaller number, |
| and | $5x =$ the larger number. |
| Then, | $25x^2 - x^2 = 96$ |
| Uniting terms, | $24x^2 = 96$ |
| Dividing by 24, | $x^2 = 4$ |
| Evolving, | $x = 2$, the smaller number. |
| Then, | $5x = 10$, the larger number. |
-
- | | |
|----------------|---|
| 3. Let | $3x =$ length of the field, |
| and | $2x =$ breadth of the field. |
| Then, | $6x^2 = 3 \text{ A. } 3 \text{ R.} = 600 \text{ rods.}$ |
| Dividing by 6, | $x^2 = 100$ |
| Evolving, | $x = 10$ |
| Then, | $3x = 30$, the length. |
| Also, | $2x = 20$, the breadth. |
-
- | | |
|--------------------------|---|
| 4. Let | $x =$ no. yards in first piece, |
| and | $y =$ no. yards in second piece. |
| Then | $x^2 =$ entire price of first, in dimes, |
| and | $y^2 =$ entire price of second, in dimes. |
| Therefore, | $x + y = 36$ (1) |
| and, | $x^2 : y^2 :: 4 : 1$ (2) |
| From (2), | $x^2 = 4y^2$ (3) |
| Evolving, | $x = 2y$ (4) |
| Substituting (4) in (1), | $2y + y = 36$ (5) |
| Uniting terms, | $3y = 36$ (6) |
| Dividing by 3, | $y = 12$ (7) |
| Substituting in (4), | $x = 2 \times 12 = 24$ (8) |

NOTE. If only one unknown quantity be used, the equation may be derived from the proportion $x^2 : (36 - x)^2 :: 4 : 1$.

5. Let $x =$ the first number,
and $y =$ the second number.

Then $xy = 750$ (1)

and $\frac{x}{y} = 3\frac{1}{3}$ (2)

From (2), $x = \frac{10y}{3}$ (3)

Substituting (3) in (1), $\frac{10y^2}{3} = 750$ (4)

Clearing of fractions, $10y^2 = 2250$ (5)

Dividing by 10, $y^2 = 225$ (6)

Evolving, $y = 15$ (7)

Substituting in (3), $x = \frac{10 \times 15}{3} = 50$ (8)

6. Let $x =$ no. men in front, at first,

and $x + 5 =$ no. men in depth, at first.

Then $x(x + 5) =$ whole number of men.

Therefore, $x(x + 5) = 5(x + 845)$

Or, $x^2 + 5x = 5x + 4225$

Whence, $x^2 = 4225$

Evolving, $x = 65$

Hence, $x(x + 5) = 4550$

7. From the conditions, $25x^2 + 49x^2 + 81x^2 = 620$

Or, $155x^2 = 620$

Dividing by 155, $x^2 = 4$

Evolving, $x = 2$

Hence, $\left. \begin{array}{l} 5x = 10 \\ 7x = 14 \\ 9x = 18 \end{array} \right\} \text{Ans.}$

8. Let x represent the number of boys, and it will also represent the number of bags taken by each, and the num-

ber of nuts each bag would hold. Therefore $x \times x = x^2$ will represent the whole number of bags, and $x^2 \times x = x^3$ the whole number of nuts.

$$\begin{array}{ll} \text{Hence,} & x^3 = 1000 \\ \text{Evolving,} & x = 10 \end{array}$$

$$\begin{array}{ll} 9. \text{ Let} & x = \text{side of the smaller,} \\ \text{and} & \frac{5x}{2} = \text{side of the larger.} \\ \text{Then} & x^3 = \text{contents of smaller,} \\ \text{and} & \frac{125x^3}{8} = \text{contents of larger.} \end{array}$$

$$\text{Therefore,} \quad \frac{125x^3}{8} - x^3 = 117$$

$$\text{Clearing of fractions,} \quad 125x^3 - 8x^3 = 936$$

$$\text{Uniting terms,} \quad 117x^3 = 936$$

$$\text{Dividing by 117,} \quad x^3 = 8$$

$$\text{Evolving,} \quad x = 2$$

$$\text{Hence,} \quad \frac{5x}{2} = 5$$

$$10. \text{ Let} \quad x = \text{A's distance,}$$

$$\text{and} \quad y = \text{B's distance.}$$

$$\text{Then} \quad \frac{y}{15\frac{1}{4}} = \frac{4y}{63} = \text{A's rate per day,}$$

$$\text{and} \quad \frac{x}{28} = \text{B's rate per day.}$$

$$\text{Also} \quad x \div \frac{4y}{63} = \frac{63x}{4y} = \text{no. days A traveled,}$$

$$\text{and} \quad y \div \frac{x}{28} = \frac{28y}{x} = \text{no. days B traveled.}$$

$$\text{From the conditions,} \quad x - y = 18 \quad (1)$$

$$\text{Also,} \quad \frac{63x}{4y} = \frac{28y}{x} \quad (2)$$

$$\text{Clearing (2) of fract.,} \quad 63x^2 = 112y^2 \quad (3)$$

$$\text{Dividing by 7,} \quad 9x^2 = 16y^2 \quad (4)$$

$$\text{Evolving,} \quad 3x = 4y \quad (5)$$

Or,	$3x - 4y = 0$	(6)
Multiplying (1) by 4,	$4x - 4y = 72$	(7)
Subtracting (6) from (7),	$x = 72$	(8)
Substituting in (1),	$72 - y = 18$	(9)
Whence,	$y = 54$	(10)

AFFECTED QUADRATIC EQUATIONS.

FIRST METHOD OF COMPLETING THE SQUARE.

(ART. 269, pp. 236-242.)

6. Given $x^2 + 2x = 8$
 Compl. square by adding $(\frac{2}{2})^2$, $x^2 + 2x + 1 = 9$
 Evolving, $x + 1 = \pm 3$
 Transposing, $x = -1 \pm 3$
 Uniting, $x = 2$, or -4

7. Given $x^2 - 4x = -4$
 Completing the square, $x^2 - 4x + 4 = 0$
 Evolving, $x - 2 = \pm 0$
 Transposing, $x = 2 \pm 0$
 Or, $x = 2$, or 2

8. Given $x^2 - 6x = 55$
 Compl. the square, $x^2 - 6x + 9 = 64$
 Evolving, $x - 3 = \pm 8$
 Transposing, $x = 3 \pm 8$
 Uniting, $x = 11$, or -5 , Ans.

9. Given $x^2 + 12x + 35 = 0$
 Transposing, $x^2 + 12x = -35$
 Completing the square, $x^2 + 12x + 36 = 1$
 Evolving, $x + 6 = \pm 1$
 Whence, $x = -6 \pm 1 = -5$, or -7

10. Given $3x^2 + 48 = 30x$
 Transposing terms, $3x^2 - 30x = -48$

- Dividing by 3, $x^2 - 10x = -16$
 Completing the square, $x^2 - 10x + 25 = 9$
 Evolving, $x - 5 = \pm 3$
 Whence, $x = 5 \pm 3 = 8, \text{ or } 2$
11. Given $x^2 - 2ax = b$
 Compl. the square, $x^2 - 2ax + a^2 = a^2 + b$
 Evolving, $x - a = \pm \sqrt{a^2 + b}$
 Transposing, $x = a \pm \sqrt{a^2 + b}$
12. Given $x^2 = 3x + 10$
 Or, $x^2 - 3x = 10$
 Compl. the square, $x^2 - 3x + \frac{9}{4} = 10 + \frac{9}{4} = \frac{49}{4}$
 Evolving, $x - \frac{3}{2} = \pm \frac{7}{2}$
 Whence, $x = \frac{3}{2} \pm \frac{7}{2} = 5, \text{ or } -2$
13. Given $2x + 60 = 2x^2$
 Or, $2x^2 - 2x = 60$
 Dividing by 2, $x^2 - x = 30$
 Compl. the square, $x^2 - x + \frac{1}{4} = 30\frac{1}{4} = \frac{121}{4}$
 Evolving, $x - \frac{1}{2} = \pm \frac{11}{2}$
 Whence, $x = \frac{1}{2} \pm \frac{11}{2} = 6, \text{ or } -5$
14. Given $4y^2 + 8y = 5$
 Dividing by 4, $y^2 + 2y = \frac{5}{4}$
 Compl. the square, $y^2 + 2y + 1 = \frac{5}{4} + 1 = \frac{9}{4}$
 Evolving, $y + 1 = \pm \frac{3}{2}$
 Whence, $y = -1 \pm \frac{3}{2} = \frac{1}{2} \text{ or } -\frac{5}{2}$
15. Given $5x^2 + 20 = 25x$
 Transp. and div. by 5, $x^2 - 5x = -4$
 Compl. the square, $x^2 - 5x + \frac{25}{4} = \frac{25}{4} - 4 = \frac{9}{4}$
 Evolving, $x - \frac{5}{2} = \pm \frac{3}{2}$
 Whence, $x = \frac{5}{2} \pm \frac{3}{2} = 4, \text{ or } 1$
16. Given $3x + 4 = 39x^{-1}$
 Multiplying by x , $3x^2 + 4x = 39$
 Dividing by 3, $x^2 + \frac{4}{3}x = 13$

$$\begin{aligned} \text{Compl. the square, } x^2 + \frac{1}{3}x + \frac{1}{36} &= 13\frac{1}{36} = 13\frac{1}{9} \\ \text{Evolving, } x + \frac{1}{6} &= \pm \sqrt{13\frac{1}{9}} \\ \text{Whence, } x &= -\frac{1}{6} \pm \sqrt{13\frac{1}{9}} = 3, \text{ or } -\frac{1}{3} \end{aligned}$$

$$\begin{aligned} 17. \text{ Given } 5x^2 - 40x &= 70 \\ \text{Dividing by 5, } x^2 - 8x &= 14 \\ \text{Completing the square, } x^2 - 8x + 16 &= 30 \\ \text{Evolving, } x - 4 &= \pm \sqrt{30} \\ \text{Transposing, } x &= 4 \pm \sqrt{30} \end{aligned}$$

$$\begin{aligned} 18. \text{ Given } 3x &= 10 + \frac{1}{4}x^2 \\ \text{Transposing, } -\frac{1}{4}x^2 + 3x &= 10 \\ \text{Mult. by } -4, x^2 - 12x &= -40 \\ \text{Compl. the square, } x^2 - 12x + 36 &= -4 + 36 = 32 \\ \text{Evolving, } x - 6 &= \pm \sqrt{32} = \pm 4\sqrt{2} \\ \text{Transposing, } x &= 6 \pm 4\sqrt{2} \end{aligned}$$

NOTE. Multiplying by -4 , is the same as dividing by $-\frac{1}{4}$, the coefficient of x^2 , according to the Rule and Note 1.

$$\begin{aligned} 19. \text{ Given } x^2 - 6x &= 0 \\ \text{Compl. the square, } x^2 - 6x + 9 &= 9 \\ \text{Evolving, } x - 3 &= \pm 3 \\ \text{Whence, } x &= 3 \pm 3 = 6, \text{ or } 0 \end{aligned}$$

NOTE. By dividing the original equation by x , and transposing, we obtain at once $x = 6$.

$$\begin{aligned} 20. \text{ Given } ax^2 + ax^{-1} &= 2a^{-1} \\ \text{Multiplying by } ax, x^3 + a^2 &= 2ax \\ \text{Transposing terms, } x^3 - 2ax &= -a^2 \\ \text{Compl. the square, } x^3 - 2ax + 1 &= 1 - a^2 \\ \text{Evolving, } x - 1 &= \pm \sqrt{1 - a^2} \\ \text{Transposing, } x &= 1 \pm \sqrt{1 - a^2} \end{aligned}$$

(ART. 270, pp. 242, 243.)

1. In this equation, $p = -8$, and $q = 9$; hence

$$x = 4 \pm \sqrt{9 + (-4)^2} = 4 \pm \sqrt{25} = 4 \pm 5 = 9, \text{ or } -1.$$

2. In this equation, $p = 16$, and $q = -55$; hence
 $x = -8 \pm \sqrt{-55 + 8^2} = -8 \pm \sqrt{9} = -8 \pm 3$
 $= -5$, or -11 .

3. In this equation, $p = -20$, and $q = 800$; hence
 $x = 10 \pm \sqrt{800 + (-10)^2} = 10 \pm \sqrt{900} = 10 \pm 30$
 $= 40$, or -20 . Ans. $x = 40$, or -20 .

4. In this equation, $p = 5$, and $q = 14$; hence
 $x = -\frac{5}{2} \pm \sqrt{14 + (\frac{5}{2})^2} = -\frac{5}{2} \pm \sqrt{\frac{81}{4}} = -\frac{5}{2} \pm \frac{9}{2}$
 $= 2$, or -7 .

5. Multiplying the given equation by 3, which is the same as dividing by $\frac{1}{3}$, the coefficient of x^2 , we have
 $x^2 + \frac{2}{3}x = 63$, in which $p = \frac{2}{3}$, and $q = 63$; hence
 $x = -\frac{2}{3} \pm \sqrt{63 + (\frac{2}{3})^2} = -\frac{2}{3} \pm \sqrt{\frac{1082}{9}} = -\frac{2}{3} \pm \frac{33}{3}$
 $= 6$, or $-10\frac{1}{3}$.

6. Given $\frac{1}{2}x^2 - \frac{1}{3}x + 7\frac{2}{3} = 8$

Multiplying by 2, $x^2 - \frac{2}{3}x + 14\frac{2}{3} = 16$

Transp. and uniting, $x^2 - \frac{2}{3}x = 1\frac{1}{3} = \frac{4}{3}$

In this equation, $p = -\frac{2}{3}$, and $q = \frac{4}{3}$; hence

$$x = \frac{1}{3} \pm \sqrt{\frac{4}{9} + \frac{4}{3}} = \frac{1}{3} \pm \sqrt{\frac{16}{3}} = \frac{2}{3} \pm \frac{4}{3} = \frac{2}{3}, \text{ or } -\frac{2}{3}.$$

SECOND METHOD OF COMPLETING THE SQUARE.

(ART. 272, pp. 243-248.)

4. Given $x^2 - 7x + 6 = 0$

Transposing, $x^2 - 7x = -6$

Multiplying by 4, $4x^2 - 28x = -24$

Completing the square,

$$4x^2 - 28x + 49 = 25$$

Evolving, $2x - 7 = \pm 5$

Transposing, $2x = 7 \pm 5 = 12$, or 2

Dividing by 2, $x = 6$, or 1

5. Given $x^2 + \frac{x}{2} = 3$
 Clearing of fractions, $2x^2 + x = 6$
 Multiplying by 8, $16x^2 + 8x = 48$
 Completing the square, $16x^2 + 8x + 1 = 49$
 Evolving, $4x + 1 = \pm 7$
 Whence, $4x = -1 \pm 7 = 6, \text{ or } -8$
 Dividing by 4, $x = 1\frac{1}{2}, \text{ or } -2$

6. Given $10x = 6x^2 + 4$
 Transposing, $-6x^2 + 10x = 4$
 Dividing by -2 , $3x^2 - 5x = -2$
 Multiplying by 12, $36x^2 - 60x = -24$
 Compl. square, $36x^2 - 60x + 25 = 1$
 Evolving, $6x - 5 = \pm 1$
 Whence, $6x = 5 \pm 1 = 6, \text{ or } 4$
 Dividing by 6, $x = 1, \text{ or } \frac{2}{3}, \text{ Ans.}$

7. Given $5x^2 = 57 - 4x$
 Transposing, $5x^2 + 4x = 57$
 Multiplying by 5, $25x^2 + 20x = 285$
 Compl. square, $25x^2 + 20x + 4 = 289$
 Evolving, $5x + 2 = \pm 17$
 Whence, $5x = -2 \pm 17 = 15, \text{ or } -19$
 Dividing by 5, $x = 3, \text{ or } -3\frac{4}{5}$

NOTE. As 4, the coefficient of x , is an *even* number, we complete the square by Note 2.

8. Given $\frac{a^2 - b^2}{c} = 2ax - cx^2$
 Transposing, $cx^2 - 2ax = \frac{b^2 - a^2}{c}$
 Multiplying by c , $c^2x^2 - 2acx = b^2 - a^2$
 Compl. the square, $c^2x^2 - 2acx + a^2 = b^2$
 Evolving, $cx - a = \pm b$
 Transposing, $cx = a \pm b$
 Dividing by c , $x = \frac{a \pm b}{c}$

NOTE. By multiplying the equation by c , we both clear of fractions and make the first term a perfect square. As $2a$ is even, we use c instead of $4c$, according to Note 2.

9. Given $\frac{x}{a} + bx^{-1} = c$
 Multiplying by ax , $x^2 + ab = acx$
 Transposing, $x^2 - acx = -ab$
 Multiplying by 4, $4x^2 - 4acx = -4ab$
 Compl. the square, $4x^2 - 4acx + a^2c^2 = a^2c^2 - 4ab$
 Evolving, $2x - ac = \pm \sqrt{a^2c^2 - 4ab}$
 Transposing, $2x = ac \pm \sqrt{a^2c^2 - 4ab}$
 Dividing by 2, $x = \frac{ac \pm \sqrt{a^2c^2 - 4ab}}{2}$
10. Given $(x + 1)(2x + 3) = 4x^2 - 22$
 Expanding, $2x^2 + 5x + 3 = 4x^2 - 22$
 Transposing and uniting, $-2x^2 + 5x = -25$
 Multiplying by -8 , $16x^2 - 40x = 200$
 Compl. the square, $16x^2 - 40x + 25 = 225$
 Evolving, $4x - 5 = \pm 15$
 Whence, $4x = 5 \pm 15 = 20$, or -10
 Dividing by 4, $x = 5$, or $-\frac{5}{2}$
11. Given $\frac{3}{4}(y^2 - 3) = \frac{1}{2}(y - 3)$
 Clearing of fractions, $6(y^2 - 3) = y - 3$
 Expanding, $6y^2 - 18 = y - 3$
 Transposing and uniting, $6y^2 - y = 15$
 Multiplying by 24, $144y^2 - 24y = 360$
 Completing the square, $144y^2 - 24y + 1 = 361$
 Evolving, $12y - 1 = \pm 19$
 Whence, $12y = 1 \pm 19 = 20$, or -18
 Dividing by 12, $y = \frac{5}{3}$, or $-\frac{3}{2}$
12. Given $x^{-2} + \frac{5}{2}x^{-1} + \frac{3}{2} = 0$

FIRST SOLUTION.

- Clearing of fractions, &c., $2x^{-2} + 5x^{-1} = -3$
 Multiplying by 8, $16x^{-2} + 40x^{-1} = -24$

Completing the square, $16x^{-2} + 40x^{-1} + 25 = 1$

Evolving, $4x^{-1} + 5 = \pm 1$

Whence, $4x^{-1} = -5 \pm 1 = -4$, or -6

Dividing by 4, $x^{-1} = -1$, or $-\frac{3}{2}$

Hence, $x = -1$, or $-\frac{2}{3}$

SECOND SOLUTION.

Multiplying by $2x^2$, $2 + 5x + 3x^2 = 0$

Transposing, $3x^2 + 5x = -2$

Multiplying by 12, $36x^2 + 60x = -24$

Completing the square, $36x^2 + 60x + 25 = 1$

Evolving, $6x + 5 = \pm 1$

Whence, $6x = -5 \pm 1 = -4$, or -6

Dividing by 6, $x = -\frac{2}{3}$, or -1

NOTE. Any of the above examples may be solved by Note 4, if the teacher prefers to use the formula there given.

THIRD METHOD OF COMPLETING THE SQUARE.

(ART. 273, pp. 248, 249.)

1. Given $9x^2 - 6x = 8$
 Compl. the square, $9x^2 - 6x + 1 = 9$
 Evolving, $3x - 1 = \pm 3$
 Whence, $3x = 1 \pm 3 = 4$, or -2
 Dividing by 3, $x = \frac{4}{3}$, or $-\frac{2}{3}$

2. Given $4x^2 + 4x = 3$
 Adding $\left(\frac{4x}{2\sqrt{4x^2}}\right)^2$, $4x^2 + 4x + 1 = 4$
 Evolving, $2x + 1 = \pm 2$
 Whence, $2x = -1 \pm 2 = 1$, or -3
 Dividing by 2, $x = \frac{1}{2}$, or $-\frac{3}{2}$, Ans.

3. Given $27ax^2 + 6bx = \frac{b^2}{a}$
 Multiplying by $3a$, $81a^2x^2 + 18abx = 3b^2$
 Compl. the square, $81a^2x^2 + 18abx + b^2 = 4b^2$
 Evolving, $9ax + b = \pm 2b$

Whence, $9ax = -b \pm 2b = b$, or $-3b$

Dividing by $9a$, $x = \frac{b}{9a}$, or $-\frac{b}{3a}$

4. Given $50x^2 + 4x = \frac{9}{10}$

Dividing by 2, $25x^2 + 2x = \frac{9}{20}$

Adding $\left(\frac{2x}{2\sqrt{25x^2}}\right)^2$, $25x^2 + 2x + \frac{1}{25} = \frac{9}{20} + \frac{1}{25} = \frac{49}{100}$

Evolving, $5x + \frac{1}{5} = \pm \frac{7}{10}$

Whence, $5x = -\frac{1}{5} \pm \frac{7}{10} = \frac{5}{10}$, or $-\frac{9}{10}$

Dividing by 5, $x = \frac{1}{10}$, or $-\frac{9}{50}$

NOTE. If we multiply by 2, instead of dividing by 2, the equation becomes $100x^2 + 8x + \frac{4}{25} = \frac{49}{25}$, after completing the square.

5. Given $5x^{-1}(5x^{-1} - 12) = -36$

Expanding, $25x^{-2} - 60x^{-1} = -36$

Adding $\left(\frac{60}{2\sqrt{25}}\right)^2$, $25x^{-2} - 60x^{-1} + 36 = 0$

Evolving, $5x^{-1} - 6 = 0$

Transposing, $5x^{-1} = 6$

Dividing by 5, $x^{-1} = \frac{6}{5}$

Hence, $x = \frac{5}{6}$

NOTE. If the unknown quantity contained in the first term is the square of that contained in the second, that unknown quantity will disappear in the third term. Hence, in such instances, we may use only the coefficients, in determining what to add in completing the square, as in the above solution.

6. Given $\frac{x^2}{16} + 4x = 17$

Adding $\left(4 \div 2\sqrt{\frac{1}{16}}\right)^2$, $\frac{x^2}{16} + 4x + 64 = 81$

Evolving, $\frac{x}{4} + 8 = \pm 9$

Whence, $\frac{x}{4} = -8 \pm 9 = 1$, or -17

Multiplying by 4, $x = 4$, or -68

7. Given $\frac{x^2}{12} + 3x = 21$
 Dividing by 3, $\frac{x^2}{36} + x = 7$
 Adding $(1 \div 2\sqrt{\frac{1}{36}})^2$, $\frac{x^2}{36} + x + 9 = 16$
 Evolving, $\frac{x}{6} + 3 = \pm 4$
 Whence, $\frac{x}{6} = -3 \pm 4 = 1, \text{ or } -7$
 Multiplying by 6, $x = 6, \text{ or } -42$

(ART. 274, pp. 250 - 252.)

1. Given $x^2 - 14x = 120$
 Adding $(\frac{14}{2})^2$, $x^2 - 14x + 49 = 169$
 Evolving, $x - 7 = \pm 13$
 Transposing, $x = 7 \pm 13$
 Uniting, $x = 20, \text{ or } -6$
2. Given $x^2 - \frac{3}{2}x = 27$
 Clearing of fractions, $2x^2 - 3x = 54$
 Multiplying by 8, $16x^2 - 24x = 432$
 Adding 3^2 , $16x^2 - 24x + 9 = 441$
 Evolving, $4x - 3 = \pm 21$
 Whence, $4x = 3 \pm 21 = 24, \text{ or } -18$
 Dividing by 4, $x = 6, \text{ or } -4\frac{1}{2}$
3. Given $2x^2 - 10x = 100$
 Dividing by 2, $x^2 - 5x = 50$
 Adding $(\frac{5}{2})^2$, $x^2 - 5x + \frac{25}{4} = 50 + \frac{25}{4} = 22\frac{1}{4}$
 Evolving, $x - \frac{5}{2} = \pm \frac{1}{2}\sqrt{22\frac{1}{4}}$
 Transposing, $x = \frac{5}{2} \pm \frac{1}{2}\sqrt{22\frac{1}{4}}$
 Uniting, $x = 10, \text{ or } -5, \text{ Ans.}$
4. Given $16x^{-1} - 4 = 12x^{-2}$
 Dividing by 4, $4x^{-1} - 1 = 3x^{-2}$
 Transposing, &c., $3x^{-2} - 4x^{-1} = -1$

$$\begin{array}{ll}
 \text{Multiplying by 3,} & 9x^2 - 12x^{-1} = -3 \\
 \text{Adding } (\frac{1}{3})^2, & 9x^2 - 12x^{-1} + 4 = 1 \\
 \text{Evolving,} & 3x^{-1} - 2 = \pm 1 \\
 \text{Whence,} & 3x^{-1} = 2 \pm 1 = 3, \text{ or } 1 \\
 \text{Dividing by 3,} & x^{-1} = 1, \text{ or } \frac{1}{3} \\
 \text{Hence,} & x = 1, \text{ or } 3
 \end{array}$$

$$\begin{array}{ll}
 5. \text{ Given} & x^3 - \frac{3}{10}x = \frac{1}{10} \\
 \text{Adding } (\frac{3}{20})^2, & x^3 - \frac{3}{10}x + \frac{9}{400} = \frac{1}{10} + \frac{9}{400} = \frac{49}{400} \\
 \text{Evolving,} & x - \frac{3}{20} = \pm \frac{7}{20} \\
 \text{Transposing,} & x = \frac{3}{20} \pm \frac{7}{20} \\
 \text{Uniting,} & x = \frac{1}{2}, \text{ or } -\frac{1}{2}
 \end{array}$$

$$\begin{array}{ll}
 6. \text{ Given} & \frac{2x^2}{3} + 3\frac{1}{2} = \frac{x}{2} + 8 \\
 \text{Multiplying by 6,} & 4x^2 + 21 = 3x + 48 \\
 \text{Transp. and uniting,} & 4x^2 - 3x = 27 \\
 \text{Multiplying by 16,} & 64x^2 - 48x = 432 \\
 \text{Adding } 3^2, & 64x^2 - 48x + 9 = 441 \\
 \text{Evolving,} & 8x - 3 = \pm 21 \\
 \text{Whence,} & 8x = 3 \pm 21 = 24, \text{ or } -18 \\
 \text{Dividing by 8,} & x = 3, \text{ or } -\frac{9}{4}, \text{ Ans.}
 \end{array}$$

$$\begin{array}{ll}
 7. \text{ Given} & \frac{z^2}{12} + \frac{20}{3} = 2z \\
 \text{Multiplying by 3,} & \frac{z^2}{4} + 20 = 6z \\
 \text{Transposing,} & \frac{z^2}{4} - 6z = -20 \\
 \text{Adding } \left(6 \div 2\sqrt{\frac{1}{4}}\right)^2, & \frac{z^2}{4} - 6z + 36 = 16 \\
 \text{Evolving,} & \frac{z}{2} - 6 = \pm 4 \\
 \text{Whence,} & \frac{z}{2} = 6 \pm 4 = 10, \text{ or } 2 \\
 \text{Multiplying by 2,} & z = 20, \text{ or } 4
 \end{array}$$

NOTE. This equation is also readily solved by the first method of completing the square.

8. Given $2x^2 + 15 = 3x$
 Transposing, $2x^2 - 3x = -15$
 Multiplying by 8, $16x^2 - 24x = -120$
 Adding 3^2 , $16x^2 - 24x + 9 = -120 + 9 = -111$
 Evolving, $4x - 3 = \pm \sqrt{-111}$
 Transposing, $4x = 3 \pm \sqrt{-111}$
 Dividing by 4, $x = \frac{3 \pm \sqrt{-111}}{4}$

NOTE. These values are imaginary.

9. Given $x^2 - 6x + 19 = 13$
 Transposing and uniting, $x^2 - 6x = -6$
 Adding $(\frac{6}{2})^2$, $x^2 - 6x + 9 = -6 + 9 = 3$
 Evolving, $x - 3 = \pm \sqrt{3}$
 Transposing, $x = 3 \pm \sqrt{3}$
 Or, $x = 3 \pm 1.732$
 Hence, $x = 4.732$, or 1.268
10. Given $4ax^2 - 2bx = c$
 Mult. by $4a$, $16a^2x^2 - 8abx = 4ac$
 Adding $(\frac{2b}{2})^2$, $16a^2x^2 - 8abx + b^2 = 4ac + b^2$
 Evolving, $4ax - b = \pm \sqrt{4ac + b^2}$
 Transposing, $4ax = b \pm \sqrt{4ac + b^2}$
 Dividing by $4a$, $x = \frac{b \pm \sqrt{4ac + b^2}}{4a}$
11. Given $x^2 - 4 = 16 - (x - 2)^2$
 Expanding, $x^2 - 4 = 16 - x^2 + 4x - 4$
 Transp. and uniting, $2x^2 - 4x = 16$
 Dividing by 2, $x^2 - 2x = 8$
 Adding $(\frac{2}{2})^2$, $x^2 - 2x + 1 = 9$
 Evolving, $x - 1 = \pm 3$
 Whence, $x = 1 \pm 3 = 4$, or -2
12. Given $(3x - 5)(2x - 5) = (x + 3)(x - 1)$
 Expanding, $6x^2 - 25x + 25 = x^2 + 2x - 3$

Transp. and uniting, $5x^2 - 27x = -28$

Mult. by 20, $100x^2 - 540x = -560$

Adding 27^2 , $100x^2 - 540x + 729 = 169$

Evolving, $10x - 27 = \pm 13$

Whence, $10x = 27 \pm 13 = 40, \text{ or } 14$

Dividing by 10, $x = 4, \text{ or } \frac{7}{5}$

13. Given $(2x - 3)^2 = 8x$

Expanding, $4x^2 - 12x + 9 = 8x$

Transp. and uniting, $4x^2 - 20x = -9$

Adding $(20 \div 2\sqrt{4})^2$, $4x^2 - 20x + 25 = 16$

Evolving, $2x - 5 = \pm 4$

Whence, $2x = 5 \pm 4 = 9, \text{ or } 1$

Dividing by 2, $x = \frac{9}{2}, \text{ or } \frac{1}{2}$

NOTE. The above equation is also readily solved by the first method of completing the square.

14. Given $\frac{1}{6}(x - 3)^2 + \frac{2}{3} = x$

Expand. and mult. by 6, $x^2 - 6x + 9 + 27 = 6x$

Transposing and uniting, $x^2 - 12x = -36$

Adding $(\frac{1}{2})^2$, $x^2 - 12x + 36 = 0$

Evolving, $x - 6 = \pm 0$

Transposing, $x = 6, \text{ or } 6$

15. Given $x^2 + (x + 1)^2 = \frac{13}{6}x(x + 1)$

Expanding, $x^2 + x^2 + 2x + 1 = \frac{13x^2 + 13x}{6}$

Clear. of fract., $6x^2 + 6x^2 + 12x + 6 = 13x^2 + 13x$

Whence, $x^2 + x = 6$

Adding $(\frac{1}{2})^2$, $x^2 + x + \frac{1}{4} = 6\frac{1}{4} = \frac{25}{4}$

Evolving, $x + \frac{1}{2} = \pm \frac{5}{2}$

Whence, $x = -\frac{1}{2} \pm \frac{5}{2} = 2, \text{ or } -3$

16. Given $3(2 - x) + 2(3 - x) = 2(4 + 3x^2)$

Expanding, $6 - 3x + 6 - 2x = 8 + 6x^2$

Whence, $6x^2 + 5x = 4$

Multiplying by 24, $144x^2 + 120x = 96$

$$\begin{array}{ll} \text{Adding } 5^2, & 144x^2 + 120x + 25 = 121 \\ \text{Evolving,} & 12x + 5 = \pm 11 \\ \text{Whence,} & 12x = -5 \pm 11 = 6, \text{ or } -16 \\ \text{Dividing by 12,} & x = \frac{1}{2}, \text{ or } -\frac{4}{3} \end{array}$$

17. Given $4(x-1) - \frac{x-1}{2x} = 3\frac{1}{2}$

Mult. by $4x$, $16x^2 - 16x - 2x + 2 = 15x$

Transp. and uniting, $16x^2 - 33x = -2$

Adding $(33 \div 2\sqrt{16})^2$,

$$16x^2 - 33x + \left(\frac{33}{8}\right)^2 = \frac{1089}{64} - 2 = \frac{961}{64}$$

Evolving,

$$4x - \frac{33}{8} = \pm \frac{31}{8}$$

Whence,

$$4x = \frac{33}{8} \pm \frac{31}{8} = 8, \text{ or } \frac{1}{4}$$

Dividing by 4,

$$x = 2, \text{ or } \frac{1}{16}$$

18. Given $\frac{6}{x+1} + \frac{2}{x} = 3$

Clear. of fractions, $6x + 2x + 2 = 3x^2 + 3x$

Whence,

$$3x^2 - 5x = 2$$

Multiplying by 12, $36x^2 - 60x = 24$

Adding 5^2 , $36x^2 - 60x + 25 = 49$

Evolving,

$$6x - 5 = \pm 7$$

Whence,

$$6x = 5 \pm 7 = 12, \text{ or } -2$$

Dividing by 6,

$$x = 2, \text{ or } -\frac{1}{3}$$

19. Given $\frac{7}{x+1} + \frac{2}{x} - 5 = 0$

Clear. of fract., $7x + 2x + 2 - 5x^2 - 5x = 0$

Whence,

$$5x^2 - 4x = 2$$

Multiplying by 5,

$$25x^2 - 20x = 10$$

Adding $(\frac{4}{5})^2$,

$$25x^2 - 20x + 4 = 14$$

Evolving,

$$5x - 2 = \pm \sqrt{14}$$

Transposing,

$$5x = 2 \pm \sqrt{14}$$

Dividing by 5,

$$x = \frac{2 \pm \sqrt{14}}{5}$$

Or,

$$x = 1.148, \text{ or } -0.348$$

20. Given $\frac{x}{x+60} = \frac{7}{3x-5}$
 Clearing of fractions, $3x^2 - 5x = 7x + 420$
 Transp. and uniting, $3x^2 - 12x = 420$
 Dividing by 3, $x^2 - 4x = 140$
 Adding $(\frac{1}{2})^2$, $x^2 - 4x + 4 = 144$
 Evolving, $x - 2 = \pm 12$
 Whence, $x = 2 \pm 12 = 14, \text{ or } -10$

21. Given $8x + 11 + 7x^{-1} = 3 + \frac{65x}{7}$
 Mult. by $7x$, $56x^2 + 77x + 49 = 21x + 65x^2$
 Whence, $9x^2 - 56x = 49$
 Adding $(\frac{56}{2 \cdot 9})^2$, $9x^2 - 56x + (\frac{28}{9})^2 = 49 + \frac{784}{9} = \frac{1225}{9}$
 Evolving, $3x - \frac{28}{9} = \pm \frac{35}{3}$
 Whence, $3x = \frac{28}{9} \pm \frac{35}{3} = 21, \text{ or } -\frac{7}{3}$
 Dividing by 3, $x = 7, \text{ or } -\frac{7}{9}$

22. Given $\frac{21}{5-x} - \frac{x}{7} = 3\frac{1}{2}$
 Clearing of fractions, $147 - 5x + x^2 = 115 - 23x$
 Transp. and uniting, $x^2 + 18x = -32$
 Adding $(\frac{18}{2})^2$, $x^2 + 18x + 81 = 49$
 Evolving, $x + 9 = \pm 7$
 Whence, $x = -9 \pm 7 = -2, \text{ or } -16$

23. Given $\frac{x^2 - 5x}{x+3} = x - 3 + x^{-1}$
 Mult. by $x(x+3)$, $x^3 - 5x^2 = x^3 - 9x + x + 3$
 Transp. and uniting, $-5x^2 + 8x = 3$
 Mult. by -5 , $25x^2 - 40x = -15$
 Adding $(\frac{8}{5})^2$, $25x^2 - 40x + 16 = 1$
 Evolving, $5x - 4 = \pm 1$
 Whence, $5x = 4 \pm 1 = 5, \text{ or } 3$
 Dividing by 5, $x = 1, \text{ or } \frac{3}{5}$

24. Given $\frac{1}{x+6} + 8x^{-1} = \frac{3}{x+2}$

Multiplying by $x(x+6)(x+2)$,

$$x^2 + 2x + 8x^2 + 64x + 96 = 3x^2 + 18x$$

Transp. and uniting, $6x^2 + 48x = -96$

Dividing by 6, $x^2 + 8x = -16$

Adding $(\frac{8}{2})^2$, $x^2 + 8x + 16 = 0$

Evolving, $x + 4 = \pm 0$

Whence, $x = -4 \pm 0 = -4$, or -4

25. Given $\frac{x+3}{x+2} + \frac{x-3}{x-2} = \frac{2x-3}{x-1}$

Clearing of fractions,

$$x^2 - 7x + 6 + x^2 - 2x^2 - 5x + 6 = 2x^2 - 3x^2 - 8x + 12$$

Transp. and uniting, $x^2 - 4x = 0$

Adding $(\frac{4}{2})^2$, $x^2 - 4x + 4 = 4$

Evolving, $x - 2 = \pm 2$

Whence, $x = 2 \pm 2 = 4$, or 0

NOTE. Dividing the equation $x^2 - 4x = 0$ by x , reduces it to a simple equation, and gives at once, $x = 4$.

26. Given $\frac{3x-2}{2x-5} + \frac{2x-5}{3x-2} = \frac{10}{3}$

Clearing of fractions,

$$3(3x-2)^2 + 3(2x-5)^2 = 10(2x-5)(3x-2)$$

Expanding,

$$27x^2 - 36x + 12 + 12x^2 - 60x + 75 = 60x^2 - 190x + 100$$

Transposing and uniting, $-21x^2 + 94x = 13$

Dividing by -21 , $x^2 - \frac{94}{21}x = -\frac{13}{21}$

Adding $(\frac{47}{21})^2$, $x^2 - \frac{94}{21}x + (\frac{47}{21})^2 = \frac{1936}{441}$

Evolving, $x - \frac{47}{21} = \pm \frac{44}{21}$

Whence, $x = \frac{47}{21} \pm \frac{44}{21} = \frac{1}{3}$, or $\frac{1}{3}$

27. Given $x^2 + ax + bx + ab = 0$

Transposing, &c., $x^2 + (a+b)x = -ab$

Adding $\left(\frac{a+b}{2}\right)^2$,

$$x^2 + (a+b)x + \left(\frac{a+b}{2}\right)^2 = \left(\frac{a+b}{2}\right)^2 - ab = \frac{a^2 - 2ab + b^2}{4}$$

Evolving, $x + \frac{a+b}{2} = \pm \frac{a-b}{2}$

Transposing, $x = \frac{-a-b}{2} \pm \frac{a-b}{2}$

Uniting, $x = -b$, or $-a$

28. Given $adx - acx^2 = bcx - bd$

Transposing, &c., $acx^2 - (ad - bc)x = bd$

Mult. by $4ac$, $4a^2c^2x^2 - 4ac(ad - bc)x = 4abcd$

Adding $(ad - bc)^2$,

$$4a^2c^2x^2 - 4ac(ad - bc)x + (ad - bc)^2 = a^2d^2 + 2abdc + b^2c^2$$

Evolving, $2acx - (ad - bc) = \pm (ad + bc)$

Transposing, $2acx = ad - bc \pm (ad + bc)$

Uniting, $2acx = 2ad$, or $-2bc$

Dividing by $2ac$, $x = \frac{d}{c}$, or $-\frac{b}{a}$

29. Given $(a+b)x^2 - cx = \frac{ac}{a+b}$

Mult. by $4(a+b)$, $4(a+b)^2x^2 - 4c(a+b)x = 4ac$

Adding c^2 , $4(a+b)^2x^2 - 4c(a+b)x + c^2 = c^2 + 4ac$

Evolving, $2(a+b)x - c = \pm \sqrt{c^2 + 4ac}$

Transposing, $2(a+b)x = c \pm \sqrt{c^2 + 4ac}$

Dividing by $2(a+b)$, $x = \frac{c \pm \sqrt{c^2 + 4ac}}{2(a+b)}$

30. Given $\sqrt{x^5} + \sqrt{x^3} = 6\sqrt{x}$

Dividing by $\sqrt{x}$, $x^2 + x = 6$

Adding $\left(\frac{1}{2}\right)^2$, $x^2 + x + \frac{1}{4} = 6\frac{1}{4} = \frac{25}{4}$

Evolving, $x + \frac{1}{2} = \pm \frac{5}{2}$

Whence, $x = -\frac{1}{2} \pm \frac{5}{2} = 2$, or -3

31. Given $(4x+5)^{\frac{1}{2}}(7x+1)^{\frac{1}{2}} = 30$

Squaring, $(4x+5)(7x+1) = 900$

$$\begin{array}{ll}
 \text{Expanding,} & 28x^2 + 39x + 5 = 900 \\
 \text{Transp. and uniting,} & 28x^2 + 39x = 895 \\
 \text{Dividing by 28,} & x^2 + \frac{39}{28}x = \frac{895}{28} \\
 \text{Adding } \left(\frac{39}{56}\right)^2, & x^2 + \frac{39}{28}x + \left(\frac{39}{56}\right)^2 = \frac{191761}{3136} \\
 \text{Evolving,} & x + \frac{39}{56} = \pm \frac{437}{56} \\
 \text{Transposing,} & x = -\frac{39}{56} \pm \frac{437}{56} \\
 \text{Uniting,} & x = 5, \text{ or } -\frac{17}{28}
 \end{array}$$

32. Given $\sqrt{2x} - 7x = -52$

$$\begin{array}{ll}
 \text{Transposing,} & \sqrt{2x} = 7x - 52 \\
 \text{Squaring,} & 2x = 49x^2 - 728x + 2704 \\
 \text{Whence,} & 49x^2 - 730x = -2704 \\
 \text{Adding } \left(\frac{730}{2 \cdot 49}\right)^2, & 49x^2 - 730x + \left(\frac{365}{7}\right)^2 = \frac{729}{49} \\
 \text{Evolving,} & 7x - \frac{365}{7} = \pm \frac{27}{7} \\
 \text{Transposing,} & 7x = \frac{365}{7} \pm \frac{27}{7} \\
 \text{Uniting,} & 7x = 56, \text{ or } \frac{338}{7} \\
 \text{Dividing by 7,} & x = 8, \text{ or } \frac{338}{49}
 \end{array}$$

33. Given $\sqrt{x+3} + \sqrt{x+8} = 5\sqrt{x}$

$$\begin{array}{ll}
 \text{Transposing,} & \sqrt{x+3} = 5\sqrt{x} - \sqrt{x+8} \\
 \text{Squaring,} & x+3 = 25x - 10\sqrt{x^2+8x} + x+8 \\
 \text{Transp. and unit.,} & 10\sqrt{x^2+8x} = 25x+5 \\
 \text{Dividing by 5,} & 2\sqrt{x^2+8x} = 5x+1 \\
 \text{Squaring,} & 4x^2+32x = 25x^2+10x+1 \\
 \text{Whence,} & 21x^2-22x=-1 \\
 \text{Mult. by 21,} & 441x^2-462x=-21 \\
 \text{Adding } (22)^2, & 441x^2-462x+121=100 \\
 \text{Evolving,} & 21x-11=\pm 10 \\
 \text{Whence,} & 21x=11\pm 10=21, \text{ or } 1 \\
 \text{Dividing by 21,} & x=1, \text{ or } \frac{1}{21}
 \end{array}$$

34. Given $\sqrt{x+4} - \sqrt{x} = \sqrt{x} + \frac{3}{2}$

$$\begin{array}{ll}
 \text{Squaring,} & x+4-2\sqrt{x^2+4x}+x=x+\frac{3}{2} \\
 \text{Transp. and uniting,} & -2\sqrt{x^2+4x}=-x-\frac{5}{2}
 \end{array}$$

$$\begin{array}{ll}
\text{Squaring,} & 4x^2 + 16x = x^2 + 5x + \frac{25}{4} \\
\text{Transp. and uniting,} & 3x^2 + 11x = \frac{25}{4} \\
\text{Multiplying by 12,} & 36x^2 + 132x = 75 \\
\text{Adding (11)^2,} & 36x^2 + 132x + 121 = 196 \\
\text{Evolving,} & 6x + 11 = \pm 14 \\
\text{Whence,} & 6x = -11 \pm 14 = 3, \text{ or } -25 \\
\text{Dividing by 6,} & x = \frac{1}{2}, \text{ or } -\frac{25}{6}
\end{array}$$

35. Given $\frac{x^2 + 1}{x^2 - 1} = x + \left(\frac{6}{x}\right)^{\frac{1}{2}}$

Dividing both numerator and denominator of the first member by $x + 1$, $\frac{x^2 - x + 1}{x - 1} = x + \left(\frac{6}{x}\right)^{\frac{1}{2}}$

Whence, $\left(\frac{6}{x}\right)^{\frac{1}{2}} = \frac{x^2 - x + 1}{x - 1} - x$

Or, $\left(\frac{6}{x}\right)^{\frac{1}{2}} = \frac{1}{x - 1}$

Squaring, $\frac{6}{x} = \frac{1}{x^2 - 2x + 1}$

Clearing of fract., $6x^2 - 12x + 6 = x$

Transp. and uniting, $6x^2 - 13x = -6$

Multiplying by 24, $144x^2 - 312x = -144$

Adding (13)^2, $144x^2 - 312x + 169 = 25$

Evolving, $12x - 13 = \pm 5$

Whence, $12x = 13 \pm 5 = 18, \text{ or } 8$

Dividing by 12, $x = \frac{3}{2}, \text{ or } \frac{2}{3}$

36. Given $\sqrt{x} + \sqrt{a - x} = \sqrt{b}$

Squaring, $x + 2\sqrt{ax - x^2} + a - x = b$

Whence, $2\sqrt{ax - x^2} = b - a$

Squaring, $4ax - 4x^2 = b^2 - 2ab + a^2$

Or, $4x^2 - 4ax = -a^2 + 2ab - b^2$

Adding $\left(\frac{4a}{2\sqrt{4}}\right)^2$, $4x^2 - 4ax + a^2 = 2ab - b^2$

Evolving, $2x - a = \pm \sqrt{2ab - b^2}$

$$\text{Transposing,} \quad 2x = a \pm \sqrt{2ab - b^2}$$

$$\text{Dividing by 2,} \quad x = \frac{a \pm \sqrt{2ab - b^2}}{2}$$

EQUATIONS IN THE QUADRATIC FORM.

(ART. 276, pp. 253, 254.)

$$\begin{aligned} 3. \quad & \text{Given} && x^4 - 9x^2 + 20 = 0 \\ & \text{Transposing,} && x^4 - 9x^2 = -20 \\ & \text{Adding } \left(\frac{9}{2}\right)^2, && x^4 - 9x^2 + \frac{81}{4} = \frac{1}{4} \\ & \text{Extracting square root,} && x^2 - \frac{9}{2} = \pm \frac{1}{2} \\ & \text{Whence,} && x^2 = \frac{9}{2} \pm \frac{1}{2} = 5, \text{ or } 4 \\ & \text{Extracting square root,} && x = \pm \sqrt{5}, \text{ or } \pm 2 \end{aligned}$$

$$\begin{aligned} 4. \quad & \text{Given} && x^6 - 35x^3 + 216 = 0 \\ & \text{Transposing,} && x^6 - 35x^3 = -216 \\ & \text{Adding } \left(\frac{35}{2}\right)^2, && x^6 - 35x^3 + \frac{1225}{4} = \frac{35}{4} \\ & \text{Extracting square root,} && x^3 - \frac{35}{2} = \pm \frac{1}{2} \\ & \text{Whence,} && x^3 = \frac{35}{2} \pm \frac{1}{2} = 27, \text{ or } 8 \\ & \text{Extracting cube root,} && x = 3, \text{ or } 2 \end{aligned}$$

$$\begin{aligned} 5. \quad & \text{Given} && 5x^6 - 90x^3 - 270 = 945 \\ & \text{Transp. and unit.,} && 5x^6 - 90x^3 = 1215 \\ & \text{Dividing by 5,} && x^6 - 18x^3 = 243 \\ & \text{Adding } \left(\frac{18}{2}\right)^2, && x^6 - 18x^3 + 81 = 324 \\ & \text{Extracting square root,} && x^3 - 9 = \pm 18 \\ & \text{Whence,} && x^3 = 9 \pm 18 = 27, \text{ or } -9 \\ & \text{Extracting cube root,} && x = 3, \text{ or } \sqrt[3]{-9} \end{aligned}$$

NOTE. As 3 is an odd number, $\sqrt[3]{-9} = -\sqrt[3]{9}$

$$\begin{aligned} 6. \quad & \text{Given} && x^{10} + 31x^5 = 32 \\ & \text{Multiplying by 4,} && 4x^{10} + 124x^5 = 128 \\ & \text{Adding } (31)^2, && 4x^{10} + 124x^5 + 961 = 1089 \\ & \text{Extracting square root,} && 2x^5 + 31 = \pm 33 \\ & \text{Whence,} && 2x^5 = -31 \pm 33 = 2, \text{ or } -64 \end{aligned}$$

Dividing by 2, $x^5 = 1$, or -32

Extracting fifth root, $x = 1$, or -2

7. Given $x^{2n} - 4x^n = 10$

Adding $(\frac{1}{2})^2$, $x^{2n} - 4x^n + 4 = 14$

Extracting square root, $x^n - 2 = \pm \sqrt{14}$

Transposing, $x^n = 2 \pm \sqrt{14}$

Extracting n th root, $x = (2 \pm \sqrt{14})^{\frac{1}{n}}$

8. Given $x^4 + 1225x^2 = 74$

FIRST SOLUTION.

Multiplying by x^2 , $x^4 + 1225 = 74x^2$

Transposing, $x^4 - 74x^2 = -1225$

Adding $(\frac{1}{2})^2$, $x^4 - 74x^2 + 1369 = 144$

Extracting square root, $x^2 - 37 = \pm 12$

Whence, $x^2 = 37 \pm 12 = 49$, or 25

Extracting square root, $x = \pm 7$, or ± 5

SECOND SOLUTION.

Adding $2(x^2)^{\frac{1}{2}}(1225x^2)^{\frac{1}{2}}$, $x^4 + 70 + 1225x^2 = 144$

Extracting square root, $x + 35x^{-1} = \pm 12$

Multiplying by x , $x^2 + 35 = \pm 12x$

Transposing, $x^2 \mp 12x = -35$

Adding $(\frac{1}{2})^2$, $x^2 \mp 12x + 36 = 1$

Extracting square root, $x \mp 6 = \pm 1$

Transposing, $x = \pm 6 \pm 1$

Uniting, $x = \pm 7$, or ± 5

(ART. 277, pp. 254-257.)

3. Given $x + 4\sqrt{x} = 21$

Adding $(\frac{1}{2})^2$, $x + 4\sqrt{x} + 4 = 25$

Extracting square root, $\sqrt{x} + 2 = \pm 5$

Whence, $\sqrt{x} = -2 \pm 5 = 3$, or -7

Squaring, $x = 9$, or 49

4. Given $x^{-1} + x^{-\frac{1}{2}} = 6$
 Adding $(\frac{1}{2})^2$, $x^{-1} + x^{-\frac{1}{2}} + \frac{1}{4} = 6\frac{1}{4} = 2\frac{5}{4}$
 Extracting square root, $x^{-\frac{1}{2}} + \frac{1}{2} = \pm \frac{5}{2}$
 Whence, $x^{-\frac{1}{2}} = -\frac{1}{2} \pm \frac{5}{2} = 2$, or -3
 Squaring, $x^{-1} = 4$, or 9
 Hence, $x = \frac{1}{4}$, or $\frac{1}{9}$

5. Given $x^{\frac{4}{3}} + 10x^{\frac{2}{3}} = 171$
 Adding $(\frac{10}{2})^2$, $x^{\frac{4}{3}} + 10x^{\frac{2}{3}} + 25 = 196$
 Extracting square root, $x^{\frac{2}{3}} + 5 = \pm 14$
 Whence, $x^{\frac{2}{3}} = -5 \pm 14 = 9$, or -19
 By Note, $x = 9^{\frac{3}{2}}$, or $(-19)^{\frac{3}{2}}$
 Or, $x = 27$, or $(-19)^{\frac{3}{2}}$

NOTE. It will be observed that the second value of x is *imaginary*.
 (Art. 250.)

6. Given $5y^{\frac{1}{2}} + y^{\frac{1}{2}} = 22$
 Multiplying by 20, $100y^{\frac{1}{2}} + 20y^{\frac{1}{2}} = 440$
 Adding 1, $100y^{\frac{1}{2}} + 20y^{\frac{1}{2}} + 1 = 441$
 Extracting square root, $10y^{\frac{1}{4}} + 1 = \pm 21$
 Whence, $10y^{\frac{1}{4}} = -1 \pm 21 = 20$, or -22
 Dividing by 10, $y^{\frac{1}{4}} = 2$, or $-\frac{11}{5}$
 Raising to fourth power, $y = 16$, or $(-\frac{11}{5})^4$

7. Given $\sqrt[5]{x} + \sqrt[5]{x^2} = 6$
 Or, $x^{\frac{2}{5}} + x^{\frac{1}{5}} = 6$
 Adding $(\frac{1}{2})^2$, $x^{\frac{2}{5}} + x^{\frac{1}{5}} + \frac{1}{4} = 6\frac{1}{4} = 2\frac{5}{4}$
 Extract. square root, $x^{\frac{1}{5}} + \frac{1}{2} = \pm \frac{5}{2}$
 Whence, $x^{\frac{1}{5}} = -\frac{1}{2} \pm \frac{5}{2} = 2$, or -3
 Raising to fifth power, $x = 32$, or -243

8. Given $x^{\frac{1}{n}} - x^{\frac{2}{n}} + 2 = 0$
 Or, $x^{\frac{2}{n}} - x^{\frac{1}{n}} = 2$
 Adding $(\frac{1}{2})^2$, $x^{\frac{2}{n}} - x^{\frac{1}{n}} + \frac{1}{4} = 2\frac{1}{4} = \frac{9}{4}$
 Extracting square root, $x^{\frac{1}{n}} - \frac{1}{2} = \pm \frac{3}{2}$
 Whence, $x^{\frac{1}{n}} = \frac{1}{2} \pm \frac{3}{2} = 2$, or -1
 Raising to n th power, $x = 2^n$, or $(-1)^n$

9. Given $x^n + p x^{\frac{n}{2}} = q$
 Adding $(\frac{p}{2})^2$, $x^n + p x^{\frac{n}{2}} + \frac{1}{4} p^2 = q + \frac{1}{4} p^2$
 Extract. sq. root, $x^{\frac{n}{2}} + \frac{1}{2} p = \pm \sqrt{q + \frac{1}{4} p^2}$
 Transposing, $x^{\frac{n}{2}} = -\frac{1}{2} p \pm \sqrt{q + \frac{1}{4} p^2}$
 By Note, $x = (-\frac{1}{2} p \pm \sqrt{q + \frac{1}{4} p^2})^{\frac{2}{n}}$

10. Given $\sqrt{x^5} - 3x = 40x^{-\frac{1}{2}}$
 Multiplying by $x^{\frac{1}{2}}$, $x^3 - 3x^{\frac{3}{2}} = 40$
 Mult. by 4, $4x^3 - 12x^{\frac{3}{2}} = 160$
 Adding 3^2 , $4x^3 - 12x^{\frac{3}{2}} + 9 = 169$
 Extract. sq. root, $2x^{\frac{3}{2}} - 3 = \pm 13$
 Whence, $2x^{\frac{3}{2}} = 3 \pm 13 = 16$, or -10
 Dividing by 2, $x^{\frac{3}{2}} = 8$, or -5
 By Note, $x = 8^{\frac{2}{3}}$, or $(-5)^{\frac{2}{3}}$
 Or, $x = 4$, or $(-5)^{\frac{2}{3}}$

(ART. 278, pp. 257-259.)

3. Given $(x-1)^2 - x = -\frac{1}{4}$
 Adding 1, $(x-1)^2 - (x-1) = \frac{3}{4}$
 Adding $(\frac{1}{2})^2$, $(x-1)^2 - (x-1) + \frac{1}{4} = 1$

Extr. sq. root, $(x - 1) - \frac{1}{2} = \pm 1$

Or, $x - \frac{3}{2} = \pm 1$

Whence, $x = \frac{3}{2} \pm 1 = 2\frac{1}{2}, \text{ or } \frac{1}{2}$

NOTE. If the given equation be expanded, it reduces to $x^2 - 3x = -\frac{1}{4}$, and this form, after completing the square and evolving, gives $x - \frac{3}{2} = \pm 1$, as above.

4. Given $(y^2 - 4y)^2 - 6(y^2 - 4y) + 5 = 0$
 Trans. and add. $(\frac{3}{2})^2$, $(y^2 - 4y)^2 - 6(y^2 - 4y) + 9 = 4$
 Extract. square root, $(y^2 - 4y) - 3 = \pm 2$
 Whence, $y^2 - 4y = 3 \pm 2 = 5, \text{ or } 1$
 Adding $(\frac{1}{2})^2$, $y^2 - 4y + 4 = 9, \text{ or } 5$
 Extract. square root, $y - 2 = \pm 3, \text{ or } \pm \sqrt{5}$
 Transposing, $y = 2 \pm 3, \text{ or } 2 \pm \sqrt{5}$
 Or, $y = 5, \text{ or } -1, \text{ or } 2 \pm \sqrt{5}$

NOTE. If we substitute x^2 for $(y^2 - 4y)^2$, and x for $(y^2 - 4y)$, the original equation becomes, $x^2 - 6x = -5$, which gives the roots $x = 5$, or 1. Replacing the value of x , we have $y^2 - 4y = 5$, or 1, as before.

5. Given $x^2 - 2x + 6\sqrt{x^2 - 2x + 5} = 11$
 Adding 5; $(x^2 - 2x + 5) + 6(x^2 - 2x + 5)^{\frac{1}{2}} = 16$
 Let $x^2 - 2x + 5 = y^2$, and $(x^2 - 2x + 5)^{\frac{1}{2}} = y$
 Then, $y^2 + 6y = 16$
 Adding $(\frac{3}{2})^2$, $y^2 + 6y + 9 = 25$
 Extracting square root, $y + 3 = \pm 5$
 Whence, $y = -3 \pm 5 = 2, \text{ or } -8$
 Squaring, $y^2 = 4, \text{ or } 64$
 Replac. val. of y^2 , $x^2 - 2x + 5 = 4, \text{ or } 64$
 Transp. and uniting, $x^2 - 2x = -1, \text{ or } 59$
 Adding $(\frac{1}{2})^2$, $x^2 - 2x + 1 = 0, \text{ or } 60$
 Extracting square root, $x - 1 = 0, \text{ or } \pm \sqrt{60}$
 Transposing, $x = 1, \text{ or } 1 \pm \sqrt{60}$
 Or, $x = 1, \text{ or } 1 \pm 2\sqrt{15}$

NOTE. The above equation can be readily solved without the use of

y. It is, however, more convenient to write a single letter in place of the three terms $x^2 - 2x + 5$, and we can also thus avoid the use of radicals.

6. Given $\sqrt{x+2} + 2\sqrt[3]{x+2} = 8$
 Adding $(\frac{2}{3})^2$, $\sqrt{x+2} + 2\sqrt[3]{x+2} + 1 = 9$
 Extracting square root, $\sqrt{x+2} + 1 = \pm 3$
 Whence, $\sqrt{x+2} = -1 \pm 3 = 2$, or -4
 Raising to fourth power, $x+2 = 16$, or 256
 Transposing and uniting, $x = 14$, or 254

7. Given $(x^2 + 7)^{\frac{5}{2}} + 2(x^2 + 7)^{\frac{3}{2}} = 80$
 Adding $(\frac{2}{3})^2$, $(x^2 + 7)^{\frac{5}{2}} + 2(x^2 + 7)^{\frac{3}{2}} + 1 = 81$
 Extracting square root, $(x^2 + 7)^{\frac{3}{2}} + 1 = \pm 9$
 Whence, $(x^2 + 7)^{\frac{3}{2}} = -1 \pm 9 = 8$, or -10
 Extracting third root, $(x^2 + 7)^{\frac{1}{2}} = 2$, or $(-10)^{\frac{1}{3}}$
 Raising to fifth power, $x^2 + 7 = 32$, or $(-10)^{\frac{5}{3}}$
 Transposing and uniting, $x^2 = 25$, or $-7 - 10^{\frac{5}{3}}$
 Extracting square root, $x = \pm 5$

NOTE. The other values of x will be *imaginary*, and as only the *real* roots are required, the imaginary ones are dropped.

8. Given $(x-3)^4 + 4(x-3)^2 = 117$
 Adding $(\frac{1}{2})^2$, $(x-3)^4 + 4(x-3)^2 + 4 = 121$
 Extracting square root, $(x-3)^2 + 2 = \pm 11$
 Whence, $(x-3)^2 = -2 \pm 11 = 9$, or -13
 Extr. square root, $x-3 = \pm 3$, or $\pm \sqrt{-13}$
 Transposing, $x = 3 \pm 3$, or $3 \pm \sqrt{-13}$
 Or, $x = 6$, or 0 , or $3 \pm \sqrt{-13}$

9. Given $\sqrt{z+12} + \sqrt[3]{z+12} = 6$
 Adding $(\frac{1}{3})^2$, $\sqrt{z+12} + \sqrt[3]{z+12} + \frac{1}{4} = 6\frac{1}{4} = 2\frac{5}{4}$

Extracting square root, $\sqrt{z+12} + \frac{1}{2} = \pm \frac{5}{2}$

Whence, $\sqrt{z+12} = -\frac{1}{2} \pm \frac{5}{2} = 2, \text{ or } -3$

Raising to fourth power, $z+12 = 16, \text{ or } 81$

Transposing and uniting, $z = 4, \text{ or } 69$

NOTE. Any of the equations in Art. 278 may be solved by substituting a single letter for the polynomial expression which is the basis of the quadratic form.

SIMULTANEOUS EQUATIONS INVOLVING QUADRATICS.

(ART. 283, pp. 260-262.)

$$\begin{array}{ll} 2. \text{ Given} & \begin{cases} x + y = 7 & (1) \\ x^2 + 2y^2 = 34 & (2) \end{cases} \end{array}$$

$$\text{From (1),} \quad x = 7 - y \quad (3)$$

$$\text{Squaring (3),} \quad x^2 = 49 - 14y + y^2 \quad (4)$$

$$\text{Subs. (4) in (2), } 49 - 14y + y^2 + 2y^2 = 34 \quad (5)$$

$$\text{Transposing and uniting, } 3y^2 - 14y = -15 \quad (6)$$

$$\text{Multiplying by 3, } 9y^2 - 42y = -45 \quad (7)$$

$$\text{Adding } (\frac{1}{2})^2, \quad 9y^2 - 42y + 49 = 4 \quad (8)$$

$$\text{Evolving,} \quad 3y - 7 = \pm 2 \quad (9)$$

$$\text{Whence,} \quad 3y = 9, \text{ or } 5$$

$$\text{Dividing by 3,} \quad y = 3, \text{ or } \frac{5}{3}$$

$$\text{Substituting in (3),} \quad x = 7 - 3, \text{ or } 7 - \frac{5}{3}$$

$$\text{Uniting,} \quad x = 4, \text{ or } \frac{16}{3}$$

$$\begin{array}{ll} 3. \text{ Given} & \begin{cases} x - \frac{x-y}{2} = 4 & (1) \\ y - \frac{x+3y}{x+2} = 1 & (2) \end{cases} \end{array}$$

$$\text{Clearing (1) of fractions, } 2x - x + y = 8 \quad (3)$$

$$\text{Clear. (2) of fract., } xy + 2y - x - 3y = x + 2 \quad (4)$$

$$\text{From (3),} \quad y = 8 - x \quad (5)$$

$$\text{From (4),} \quad xy - 2x - y = 2 \quad (6)$$

$$\text{Subs. (5) in (6), } x(8-x) - 2x - (8-x) = 2 \quad (7)$$

Or, $8x - x^2 - 2x - 8 + x = 2$ (8)

Whence, $x^2 - 7x = -10$ (9)

Adding $(\frac{7}{2})^2$, $x^2 - 7x + \frac{49}{4} = \frac{9}{4}$ (10)

Evolving, $x - \frac{7}{2} = \pm \frac{3}{2}$ (11)

Whence, $x = 5$, or 2

Substituting in (5), $y = 8 - 5$, or $8 - 2$

Uniting, $y = 3$, or 6

4. Given $\begin{cases} x + 4y = 23 & (1) \\ x^2 + 3xy = 54 & (2) \end{cases}$

From (1), $y = \frac{23 - x}{4}$ (3)

Subs. (3) in (2), $x^2 + 3x \left(\frac{23 - x}{4} \right) = 54$ (4)

Exp. and mult. by 4, $4x^2 + 69x - 3x^2 = 216$ (5)

Uniting terms, $x^2 + 69x = 216$ (6)

Adding $(\frac{69}{2})^2$, $x^2 + 69x + \frac{4761}{4} = \frac{5625}{4}$ (7)

Evolving, $x + \frac{69}{2} = \pm \frac{75}{2}$ (8)

Whence, $x = 3$, or -72

Subs. in (3), $y = \frac{23 - 3}{4}$, or $\frac{23 + 72}{4}$

Whence, $y = 5$, or $\frac{95}{4}$

5. Given $\begin{cases} 49x^2 = 36y^2 & (1) \\ x(2x + \frac{1}{2}) + 3xy - y(6y + 5) + 128 = 0 & (2) \end{cases}$

Extracting square root of (1), $7x = 6y$ (3)

Expanding (2), $2x^2 + \frac{1}{2}x + 3xy - 6y^2 - 5y = -128$ (4)

From (3), $y = \frac{7x}{6}$ (5)

Subs. (5) in (4), $2x^2 + \frac{1}{2}x + \frac{7x^2}{2} - \frac{49x^2}{6} - \frac{35x}{6} = -128$ (6)

Mult. by 6, $12x^2 + 3x + 21x^2 - 49x^2 - 35x = -768$ (7)

Uniting terms, $-16x^2 - 32x = -768$ (8)

Dividing by -16 , $x^2 + 2x = 48$ (9)

Adding $(\frac{2}{2})^2$, $x^2 + 2x + 1 = 49$ (10)

Evolving, $x + 1 = \pm 7$ (11)

Whence,

$$x = 6, \text{ or } -8$$

Substituting in (5),

$$y = \frac{7 \times 6}{6}, \text{ or } \frac{-8 \times 7}{6}$$

Whence,

$$y = 7, \text{ or } -\frac{28}{3}$$

(ART. 284, pp. 262, 263.)

$$2. \text{ Given } \begin{cases} y^2 - x^2 = 3 & (1) \\ y^2 - 2xy + 2x^2 = 2 & (2) \end{cases}$$

$$\text{Let } y = vx \quad (3)$$

$$\text{Subs. in (1), } v^2 x^2 - x^2 = 3 \quad (4)$$

$$\text{Subs. in (2), } v^2 x^2 - 2vx^2 + 2x^2 = 2 \quad (5)$$

$$\text{From (4), } x^2 = \frac{3}{v^2 - 1} \quad (6)$$

$$\text{From (5), } x^2 = \frac{2}{v^2 - 2v + 2} \quad (7)$$

$$\text{Hence, } \frac{3}{v^2 - 1} = \frac{2}{v^2 - 2v + 2} \quad (8)$$

$$\text{Clear. of fractions, } 3v^2 - 6v + 6 = 2v^2 - 2 \quad (9)$$

$$\text{Transposing and uniting, } v^2 - 6v = -8 \quad (10)$$

$$\text{Adding } \left(\frac{3}{2}\right)^2, \quad v^2 - 6v + 9 = 1 \quad (11)$$

$$\text{Evolving, } v - 3 = \pm 1 \quad (12)$$

$$\text{Whence, } v = 4, \text{ or } 2$$

$$\text{Substituting in (6), } x^2 = \frac{3}{16 - 1}, \text{ or } \frac{3}{4 - 1}$$

$$\text{Reducing, } x^2 = \frac{1}{5}, \text{ or } 1$$

$$\text{Evolving, } x = \pm \sqrt{\frac{1}{5}}, \text{ or } \pm 1$$

$$\text{Or, } x = \pm \frac{1}{5} \sqrt{5}, \text{ or } \pm 1$$

$$\text{Substituting in (3), } y = \pm \frac{1}{5} \sqrt{5} \times 4, \text{ or } \pm 1 \times 2$$

$$\text{Reducing, } y = \pm \frac{4}{5} \sqrt{5}, \text{ or } \pm 2$$

$$3. \text{ Given } \begin{cases} x^2 + 3xy - y^2 = 27 & (1) \\ 3x^2 + 2xy = 63 & (2) \end{cases}$$

$$\text{Let } y = vx \quad (3)$$

$$\text{Subs. in (1), } x^2 + 3vx^2 - v^2 x^2 = 27 \quad (4)$$

$$\text{Subs. in (2), } 3x^2 + 2vx^2 = 63 \quad (5)$$

From (4),
$$x^2 = \frac{27}{1 + 3v - v^2} \quad (6)$$

From (5),
$$x^2 = \frac{63}{3 + 2v} \quad (7)$$

Hence,
$$\frac{27}{1 + 3v - v^2} = \frac{63}{3 + 2v} \quad (8)$$

Dividing by 9,
$$\frac{3}{1 + 3v - v^2} = \frac{7}{3 + 2v} \quad (9)$$

Clearing of fractions,
$$9 + 6v = 7 + 21v - 7v^2 \quad (10)$$

Tr. and uniting,
$$7v^2 - 15v = -2 \quad (11)$$

Mult. by 28,
$$196v^2 - 420v = -56 \quad (12)$$

Adding (15)²,
$$196v^2 - 420v + 225 = 169 \quad (13)$$

Evolving,
$$14v - 15 = \pm 13 \quad (14)$$

Whence,
$$14v = 28, \text{ or } 2 \quad (15)$$

Dividing by 14,
$$v = 2, \text{ or } \frac{1}{7}$$

Substituting in (7),
$$x^2 = \frac{63}{3 + 4}, \text{ or } \frac{63}{3 + \frac{1}{7}}$$

Reducing,
$$x^2 = 9, \text{ or } \frac{441}{23}$$

Evolving,
$$x = \pm 3, \text{ or } \pm \frac{21}{\sqrt{23}}$$

Or,
$$x = \pm 3, \text{ or } \pm \frac{21}{\sqrt{23}}$$

Substituting in (3),
$$y = \pm 3 \times 2, \text{ or } \pm \frac{21}{\sqrt{23}} \times \frac{1}{7}$$

Reducing,
$$y = \pm 6, \text{ or } \pm \frac{3}{\sqrt{23}}$$

The last two examples may be solved in accordance with the principles contained in the note (Art. 284, Alg.), as follows:

2. Given
$$\begin{cases} y^2 - x^2 = 3 & (1) \\ y^2 - 2xy + 2x^2 = 2 & (2) \end{cases}$$

From (1),
$$y^2 = 3 + x^2 \quad (3)$$

Evolving,
$$y = \pm \sqrt{3 + x^2} \quad (4)$$

Subst. in (2),
$$3 + x^2 \mp 2x\sqrt{3 + x^2} + 2x^2 = 2 \quad (5)$$

Transp. and uniting,
$$3x^2 + 1 = \pm 2x\sqrt{3 + x^2} \quad (6)$$

Squaring,
$$9x^4 + 6x^2 + 1 = 12x^2 + 4x^4 \quad (7)$$

Transp. and uniting,
$$5x^4 - 6x^2 = -1 \quad (8)$$

Multiplying by 5, $25x^4 - 30x^2 = -5$ (9)

Adding $(\frac{3}{2})^2$, $25x^4 - 30x^2 + 9 = 4$

Evolving, $5x^2 - 3 = \pm 2$

Whence, $5x^2 = 5$, or 1

Dividing by 5, $x^2 = 1$, or $\frac{1}{5}$

Evolving, $x = \pm 1$, or $\pm \frac{1}{5}\sqrt{5}$

Substituting in (4), $y = \pm \sqrt{3+1}$, or $\pm \sqrt{3+\frac{1}{5}}$

Reducing, $y = \pm 2$, or $\pm \frac{4}{5}\sqrt{5}$

NOTE. Equation (8) is also readily obtained as follows: Subtract (2) from (1), and the result gives $y = \frac{3x^2+1}{2x}$, which, after squaring, may be substituted directly in (1).

3. Given $\begin{cases} x^2 + 3xy - y^2 = 27 & (1) \\ 3x^2 + 2xy = 63 & (2) \end{cases}$

From (2), $y = \frac{63 - 3x^2}{2x}$ (3)

Squaring, $y^2 = \frac{3969 - 378x^2 + 9x^4}{4x^2}$ (4)

Subs. in (1), $x^2 + \frac{189 - 9x^2}{2} - \frac{3969 - 378x^2 + 9x^4}{4x^2} = 27$ (5)

Clearing of fractions,

$4x^4 + 378x^2 - 18x^4 - 3969 + 378x^2 - 9x^4 = 108x^2$ (6)

Whence, $23x^4 - 648x^2 = -3969$ (7)

Multiplying by 23, $529x^4 - 14904x^2 = -91287$ (8)

Adding $(\frac{324}{23})^2$, $529x^4 - 14904x^2 + (324)^2 = 13689$

Evolving, $23x^2 - 324 = \pm 117$

Whence, $23x^2 = 441$, or 207

Dividing by 23, $x^2 = \frac{441}{23}$, or 9

Evolving, $x = \pm \frac{21}{\sqrt{23}}$, or ± 3

Substituting in (3), $y = \frac{63 - \frac{1323}{23}}{\pm \frac{42}{\sqrt{23}}}$, or $\frac{63 - 27}{\pm 6}$

Reducing, $y = \pm \frac{3}{\sqrt{23}}$, or ± 6

NOTE. When both equations contain both powers of each letter, this method of elimination becomes rather tedious. In such a case we should first eliminate the square of one of the letters, and proceed as in the preceding note. It will be seen that y has but one power in one equation of each of the examples given.

(ART. 285, pp. 264–267.)

4. Given	$\begin{cases} x + y = 4 & (1) \\ x^{-1} + y^{-1} = 1 & (2) \end{cases}$
Multiplying (2) by xy ,	$y + x = xy \quad (3)$
From (1) and (3),	$xy = 4 \quad (4)$
Squaring (1),	$x^2 + 2xy + y^2 = 16 \quad (5)$
Multiplying (4) by 4,	$4xy = 16 \quad (6)$
Subtracting (6) from (5),	$x^2 - 2xy + y^2 = 0 \quad (7)$
Evolving,	$x - y = 0 \quad (8)$
Equation (1),	$x + y = 4$
Adding (8) and (1),	$2x = 4$
Whence,	$x = 2$
Subtracting (8) from (1),	$2y = 4$
Whence,	$y = 2$

5. Given	$\begin{cases} x^2 + y^2 = 65 & (1) \\ x + y = 5 & (2) \end{cases}$
Dividing (1) by (2),	$x^2 - xy + y^2 = 13 \quad (3)$
Squaring (2),	$x^2 + 2xy + y^2 = 25 \quad (4)$
Subtracting (3) from (4),	$3xy = 12 \quad (5)$
Dividing (5) by 3,	$xy = 4 \quad (6)$
Subtracting (6) from (3),	$x^2 - 2xy + y^2 = 9 \quad (7)$
Evolving,	$x - y = \pm 3 \quad (8)$
Equation (2),	$x + y = 5$
Adding (8) and (2),	$2x = 8, \text{ or } 2$
Whence,	$x = 4, \text{ or } 1$
Subtracting (8) from (2),	$2y = 2, \text{ or } 8$
Whence,	$y = 1, \text{ or } 4$

6. Given

$$\begin{cases} \frac{x^2}{y} + \frac{y^2}{x} = 9 & (1) \\ x + y = 6 & (2) \end{cases}$$

Clearing (1) of fractions,

$$x^3 + y^3 = 9xy \quad (3)$$

Dividing (3) by (2),

$$x^2 - xy + y^2 = \frac{3}{2}xy \quad (4)$$

Transposing and uniting,

$$x^2 - \frac{5}{2}xy + y^2 = 0 \quad (5)$$

Squaring (2),

$$x^2 + 2xy + y^2 = 36 \quad (6)$$

Subtracting (5) from (6),

$$\frac{3}{2}xy = 36 \quad (7)$$

Whence,

$$xy = 8 \quad (8)$$

Multiplying by 4,

$$4xy = 32 \quad (9)$$

Subtracting (9) from (6),

$$x^2 - 2xy + y^2 = 4 \quad (10)$$

Evolving,

$$x - y = \pm 2 \quad (11)$$

Equation (2),

$$x + y = 6$$

Adding (11) and (2),

$$2x = 8, \text{ or } 4$$

Whence,

$$x = 4, \text{ or } 2$$

Subtracting (11) from (2),

$$2y = 4, \text{ or } 8$$

Whence,

$$y = 2, \text{ or } 4$$

7. Given

$$\begin{cases} x^4 + x^2y^2 + y^4 = 931 & (1) \\ x^2 + xy + y^2 = 49 & (2) \end{cases}$$

Dividing (1) by (2),

$$x^2 - xy + y^2 = 19 \quad (3)$$

Subtracting (3) from (2),

$$2xy = 30 \quad (4)$$

Dividing by 2,

$$xy = 15 \quad (5)$$

Adding (5) to (2),

$$x^2 + 2xy + y^2 = 64 \quad (6)$$

Subtracting (5) from (3),

$$x^2 - 2xy + y^2 = 4 \quad (7)$$

Extracting square root of (6),

$$x + y = \pm 8 \quad (8)$$

Extracting square root of (7),

$$x - y = \pm 2 \quad (9)$$

Adding (9) and (8),

$$2x = \pm 10, \text{ or } \pm 6$$

Whence,

$$x = \pm 5, \text{ or } \pm 3$$

Subtracting (9) from (8),

$$2y = \pm 6, \text{ or } \pm 10$$

Whence,

$$y = \pm 3, \text{ or } \pm 5$$

NOTE. It will be seen that the original equations, after dividing one by the other, are homogeneous, and of the second degree; hence the ex-

ample might be classed under Case II. The double signs of the roots also indicate the nature of their equations. (Alg., Art. 284, Note.)

8. Given

$$\begin{cases} x + y = 61 & (1) \end{cases}$$

$$\begin{cases} x^{\frac{1}{2}} + y^{\frac{1}{2}} = 11 & (2) \end{cases}$$

FIRST SOLUTION.

$$\text{Substituting } v \text{ for } x^{\frac{1}{2}} \text{ and } z \text{ for } y^{\frac{1}{2}}, \quad \begin{cases} v^2 + z^2 = 61 & (3) \\ v + z = 11 & (4) \end{cases}$$

$$\text{Squaring (4),} \quad v^2 + 2vz + z^2 = 121 \quad (5)$$

$$\text{Subtracting (3) from (5),} \quad 2vz = 60 \quad (6)$$

$$\text{Subtracting (6) from (3),} \quad v^2 - 2vz + z^2 = 1 \quad (7)$$

$$\text{Evolving,} \quad v - z = \pm 1 \quad (8)$$

$$\text{Equation (4),} \quad v + z = 11$$

$$\text{Adding (8) and (4),} \quad 2v = 12, \text{ or } 10$$

$$\text{Whence,} \quad v = 6, \text{ or } 5$$

$$\text{Subtracting (8) from (4),} \quad 2z = 10, \text{ or } 12$$

$$\text{Whence,} \quad z = 5, \text{ or } 6$$

$$\text{Replacing } x^{\frac{1}{2}} \text{ and } y^{\frac{1}{2}}, \quad x^{\frac{1}{2}} = 6, \text{ or } 5, \text{ and } y^{\frac{1}{2}} = 5, \text{ or } 6$$

$$\text{Squaring,} \quad x = 36, \text{ or } 25, \text{ and } y = 25, \text{ or } 36$$

SECOND SOLUTION.

$$\text{Squaring (2),} \quad x + 2x^{\frac{1}{2}}y^{\frac{1}{2}} + y = 121 \quad (3)$$

$$\text{Equation (1),} \quad x + y = 61$$

$$\text{Subtracting (1) from (3),} \quad 2x^{\frac{1}{2}}y^{\frac{1}{2}} = 60 \quad (4)$$

$$\text{Squaring (4),} \quad 4xy = 3600 \quad (5)$$

$$\text{Squaring (1),} \quad x^2 + 2xy + y^2 = 3721 \quad (6)$$

$$\text{Subtracting (5) from (6),} \quad x^2 - 2xy + y^2 = 121 \quad (7)$$

$$\text{Evolving,} \quad x - y = \pm 11 \quad (8)$$

$$\text{Equation (1),} \quad x + y = 61$$

$$\text{Adding (8) and (1),} \quad 2x = 72, \text{ or } 50$$

$$\text{Whence,} \quad x = 36, \text{ or } 25$$

$$\text{Subtracting (8) from (1),} \quad 2y = 50, \text{ or } 72$$

$$\text{Whence,} \quad y = 25, \text{ or } 36$$

(ART. 286, pp. 267, 268.)

$$\begin{array}{l} 1. \text{ Given } \left\{ \begin{array}{l} x^2 + y^2 + xy - 2x - 2y = 9 \\ xy = 6 \end{array} \right. \end{array} \quad \begin{array}{l} (1) \\ (2) \end{array}$$

$$\text{Adding (1) and (2), } (x+y)^2 - 2(x+y) = 15 \quad (3)$$

$$\text{Adding } \left(\frac{1}{2}\right)^2, \quad (x+y)^2 - 2(x+y) + 1 = 16 \quad (4)$$

$$\text{Evolving,} \quad x+y-1 = \pm 4 \quad (5)$$

$$\text{Whence,} \quad x+y = 5, \text{ or } -3 \quad (6)$$

$$\text{Squaring (6),} \quad x^2 + 2xy + y^2 = 25 \quad (7)$$

$$\text{Multiplying (2) by 4,} \quad 4xy = 24 \quad (8)$$

$$\text{Subtracting (8) from (7),} \quad x^2 - 2xy + y^2 = 1 \quad (9)$$

$$\text{Evolving,} \quad x-y = \pm 1 \quad (10)$$

$$\text{Equation (6),} \quad x+y = 5$$

$$\text{Adding (10) and (6),} \quad 2x = 6, \text{ or } 4$$

$$\text{Whence,} \quad x = 3, \text{ or } 2$$

$$\text{Subtracting (10) from (6),} \quad 2y = 4, \text{ or } 6$$

$$\text{Whence,} \quad y = 2, \text{ or } 3$$

$$\begin{array}{l} 2. \text{ Given } \left\{ \begin{array}{l} 4xy = 96 - x^2y^2 \\ x+y = 6 \end{array} \right. \end{array} \quad \begin{array}{l} (1) \\ (2) \end{array}$$

$$\text{From (1),} \quad x^2y^2 + 4xy + 4 = 100 \quad (3)$$

$$\text{Evolving,} \quad xy + 2 = \pm 10 \quad (4)$$

$$\text{Whence,} \quad xy = 8, \text{ or } -12 \quad (5)$$

$$\text{Squaring (2),} \quad x^2 + 2xy + y^2 = 36 \quad (6)$$

$$\text{Multiplying (5) by 4,} \quad 4xy = 32, \text{ or } -48 \quad (7)$$

$$\text{Subtracting (7) from (6),} \quad x^2 - 2xy + y^2 = 4, \text{ or } 84 \quad (8)$$

$$\text{Evolving,} \quad x-y = \pm 2, \text{ or } \pm 2\sqrt{21} \quad (9)$$

$$\text{Equation (2),} \quad x+y = 6$$

$$\text{Adding (9) and (2),} \quad 2x = 8, \text{ or } 4, \text{ or } 6 \pm 2\sqrt{21}$$

$$\text{Whence,} \quad x = 4, \text{ or } 2, \text{ or } 3 \pm \sqrt{21}$$

$$\text{Subtracting (9) from (2),} \quad 2y = 4, \text{ or } 8, \text{ or } 6 \mp 2\sqrt{21}$$

$$\text{Whence,} \quad y = 2, \text{ or } 4, \text{ or } 3 \mp \sqrt{21}$$

$$3. \text{ Given } \begin{cases} \frac{x^2}{y^2} + \frac{4x}{y} = \frac{85}{9} \\ x - y = 2 \end{cases} \quad (1)$$

$$\text{Adding } \left(\frac{4}{2}\right)^2 \text{ to (1), } \frac{x^2}{y^2} + \frac{4x}{y} + 4 = \frac{121}{9} \quad (3)$$

$$\text{Evolving, } \frac{x}{y} + 2 = \pm \frac{11}{3} \quad (4)$$

$$\text{Transposing, } \frac{x}{y} = -2 \pm \frac{11}{3} \quad (5)$$

$$\text{Uniting, } \frac{x}{y} = \frac{5}{3}, \text{ or } -\frac{17}{3} \quad (6)$$

$$\text{Whence, } x = \frac{5y}{3}, \text{ or } -\frac{17y}{3} \quad (7)$$

$$\text{Substituting in (2), } \frac{5y}{3} - y = 2, \text{ or } -\frac{17y}{3} - y = 2$$

$$\text{Whence, } 2y = 6, \text{ or } -20y = 6$$

$$\text{Therefore, } y = 3, \text{ or } -\frac{3}{10}$$

$$\text{Substituting in (7), } x = \frac{5 \times 3}{3}, \text{ or } \left(-\frac{17}{3}\right) \times \left(-\frac{3}{10}\right)$$

$$\text{Reducing, } x = 5, \text{ or } \frac{17}{10}$$

(ART. 287, pp. 268, 269.)

$$1. \text{ Given } \begin{cases} x^2 + xy = 60 \\ y^2 + xy = 84 \end{cases} \quad (1)$$

$$\text{Adding (1) and (2), } x^2 + 2xy + y^2 = 144 \quad (3)$$

$$\text{Evolving, } x + y = \pm 12 \quad (4)$$

$$\text{Dividing (1) by (4), } x = \pm 5$$

$$\text{Dividing (2) by (4), } y = \pm 7$$

$$2. \text{ Given } \begin{cases} x^2 + 9y^2 = 52 \\ x + 3y = 10 \end{cases} \quad (1)$$

$$\text{Squaring (2), } x^2 + 6xy + 9y^2 = 100 \quad (3)$$

$$\text{Subtracting (1) from (3), } 6xy = 48 \quad (4)$$

$$\text{Subtracting (4) from (1), } x^2 - 6xy + 9y^2 = 4 \quad (5)$$

$$\text{Evolving, } x - 3y = \pm 2 \quad (6)$$

$$\text{Equation (2), } x + 3y = 10$$

Adding (6) and (2), $2x = 12$, or 8

Whence, $x = 6$, or 4

Subtracting (6) from (2), $6y = 8$, or 12

Whence, $y = \frac{4}{3}$, or 2

NOTE. We take precisely the same steps that we should take if the equations were $x^2 + z^2 = 52$ and $x + z = 10$.

3. Given
$$\begin{cases} x^{\frac{2}{3}} + y^{\frac{1}{2}} = 7 & (1) \\ x^{\frac{1}{3}} y = 144 & (2) \end{cases}$$

Substituting v for $x^{\frac{2}{3}}$
$$\begin{cases} v + z = 7 & (3) \end{cases}$$

and z for $y^{\frac{1}{2}}$,
$$\begin{cases} v^2 z^2 = 144 & (4) \end{cases}$$

Extracting square root of (4), $vz = \pm 12$ (5)

Squaring (3), $v^2 + 2vz + z^2 = 49$ (6)

Multiplying (5) by 4, $4vz = \pm 48$ (7)

Subtract. (7) from (6), $v^2 - 2vz + z^2 = 1$, or 97 (8)

Evolving, $v - z = \pm 1$, or $\pm \sqrt{97}$ (9)

Equation (3), $v + z = 7$

Adding (9) and (3), $2v = 8$, or 6, or $7 \pm \sqrt{97}$

Whence, $x^{\frac{2}{3}} = v = 4$, or 3, or $\frac{7 \pm \sqrt{97}}{2}$

Transposing exponent (Art. 277, Note),

$$x = 4^{\frac{3}{2}}$$
, or $3^{\frac{3}{2}}$, or $\left(\frac{7 \pm \sqrt{97}}{2}\right)^{\frac{3}{2}}$

Or, $x = \pm 8$, or $\pm 3\sqrt{3}$, or $\left(\frac{7 \pm \sqrt{97}}{2}\right)^{\frac{3}{2}}$

Subt. (9) from (3), $2z = 6$, or 8, or $7 \mp \sqrt{97}$

Whence, $y^{\frac{1}{2}} = z = 3$, or 4, or $\frac{7 \mp \sqrt{97}}{2}$

Squaring, $y = 9$, or 16, or $\left(\frac{7 \mp \sqrt{97}}{2}\right)^2$

The solution given above, expressed without the use of v and z , would be

$$\text{Extracting square root of (2), } x^{\frac{3}{2}} y^{\frac{1}{2}} = \pm 12 \quad (3)$$

$$\text{Squaring (1), } x^{\frac{4}{3}} + 2 x^{\frac{3}{2}} y^{\frac{1}{2}} + y = 49 \quad (4)$$

$$\text{Multiplying (3) by 4, } 4 x^{\frac{3}{2}} y^{\frac{1}{2}} = \pm 48 \quad (5)$$

$$\text{Subt. (5) from (4), } x^{\frac{4}{3}} - 2 x^{\frac{3}{2}} y^{\frac{1}{2}} + y = 1, \text{ or } 97 \quad (6)$$

$$\text{Evolving, } x^{\frac{2}{3}} - y^{\frac{1}{2}} = \pm 1, \text{ or } \pm \sqrt{97} \quad (7)$$

$$\text{Equation (1), } x^{\frac{2}{3}} + y^{\frac{1}{2}} = 7$$

$$\text{Adding (7) and (1), } 2 x^{\frac{2}{3}} = 8, \text{ or } 6, \text{ or } 7 \pm \sqrt{97}$$

$$\text{Subtracting (7) from (1), } 2 y^{\frac{1}{2}} = 6, \text{ or } 8, \text{ or } 7 \mp \sqrt{97}$$

$$\begin{array}{l} 4. \text{ Given} \\ \left\{ \begin{array}{l} x^2 + 3xy = 54 \\ xy + 4y^2 = 115 \end{array} \right. \end{array} \quad \begin{array}{l} (1) \\ (2) \end{array}$$

$$\text{Adding (1) and (2), } x^2 + 4xy + 4y^2 = 169 \quad (3)$$

$$\text{Extracting square root of (3), } x + 2y = \pm 13 \quad (4)$$

$$\text{From (4), } x = \pm 13 - 2y \quad (5)$$

$$\text{Subs. (5) in (2), } \pm 13y - 2y^2 + 4y^2 = 115 \quad (6)$$

$$\text{Uniting terms, } 2y^2 \pm 13y = 115 \quad (7)$$

$$\text{Dividing by 2, } y^2 \pm \frac{13}{2}y = \frac{115}{2} \quad (8)$$

$$\text{Adding } (\frac{13}{4})^2, \quad y^2 \pm \frac{13}{2}y + \frac{169}{16} = \frac{1922}{16} \quad (9)$$

$$\text{Evolving, } y \pm \frac{13}{4} = \pm \frac{22}{4} \quad (10)$$

$$\text{Transposing, } y = \mp \frac{13}{4} \pm \frac{22}{4} \quad (11)$$

$$\text{Uniting, } y = \pm 5, \text{ or } \mp \frac{23}{2} \quad (12)$$

$$\text{Substituting in (5), } x = \pm 13 \mp 10, \text{ or } \pm 13 \pm 23 \quad (13)$$

$$\text{Uniting, } x = \pm 3, \text{ or } \pm 36 \quad (14)$$

NOTE. In equation (11), the signs $\mp$ and $\pm$ are entirely independent of each other, as they originate in different evolutions; hence they produce four distinct values of y , thus: $y = -\frac{13}{4} + \frac{22}{4}$, or $\frac{13}{4} - \frac{22}{4}$, or $-\frac{13}{4} - \frac{22}{4}$, or $\frac{13}{4} + \frac{22}{4}$. But the double signs of equation (13) are not all independent in their origin, as the sign of $2y$ is dependent upon that of 13; hence we do not here obtain four values from two double signs.

It is evident, from the form of the equations, and also of their roots, that the above example might be classed under Case II.

(ART. 288, pp. 269-271.)

1. Given	$\begin{cases} x^2 - y^2 = 24 \\ x + y = 6 \end{cases}$	(1)
	$x - y = 4$	(2)
Dividing (1) by (2),	$x - y = 4$	(3)
Adding (2) and (3),	$2x = 10$	(4)
Whence,	$x = 5$	(5)
Subtracting (3) from (2),	$2y = 2$	(6)
Whence,	$y = 1$	(7)

2. Given	$\begin{cases} x + y = a \\ x y = b \end{cases}$	(1)
	$x y = b$	(2)
Squaring (1),	$x^2 + 2xy + y^2 = a^2$	(3)
Mult. (2) by 4,	$4xy = 4b$	(4)
Subt. (4) from (3),	$x^2 - 2xy + y^2 = a^2 - 4b$	(5)
Evolving,	$x - y = \pm \sqrt{a^2 - 4b}$	(6)
Equation (1),	$x + y = a$	
Adding (6) and (1),	$2x = a \pm \sqrt{a^2 - 4b}$	
Whence,	$x = \frac{a \pm \sqrt{a^2 - 4b}}{2}$	
Subtracting (6) from (1),	$2y = a \mp \sqrt{a^2 - 4b}$	
Whence,	$y = \frac{a \mp \sqrt{a^2 - 4b}}{2}$	

3. Given	$\begin{cases} x + y = a \\ x^2 + y^2 = c \end{cases}$	(1)
	$x^2 + y^2 = c$	(2)
Squaring (1),	$x^2 + 2xy + y^2 = a^2$	(3)
Subt. (2) from (3),	$2xy = a^2 - c$	(4)
Subt. (4) from (2),	$x^2 - 2xy + y^2 = 2c - a^2$	(5)
Evolving,	$x - y = \pm \sqrt{2c - a^2}$	(6)
Equation (1),	$x + y = a$	
Adding (6) and (1),	$2x = a \pm \sqrt{2c - a^2}$	
Whence,	$x = \frac{a \pm \sqrt{2c - a^2}}{2}$	

Subtracting (6) from (1),

$$2y = a \mp \sqrt{2c - a^2}$$

Whence,

$$y = \frac{a \mp \sqrt{2c - a^2}}{2}$$

NOTE. Equation (5) may also be obtained by subtracting (3) from (2) multiplied by 2.

4. Given

$$\begin{cases} x + y = 2 & (1) \\ x^{-1} + y^{-1} = 2 & (2) \end{cases}$$

Multiplying (2) by xy ,

$$y + x = 2xy \quad (3)$$

From (1) and (3),

$$2xy = 2 \quad (4)$$

Squaring (1),

$$x^2 + 2xy + y^2 = 4 \quad (5)$$

Multiplying (4) by 2,

$$4xy = 4 \quad (6)$$

Subtracting (6) from (5),

$$x^2 - 2xy + y^2 = 0 \quad (7)$$

Evolving,

$$x - y = 0 \quad (8)$$

Equation (1),

$$x + y = 2$$

Adding (8) and (1),

$$2x = 2$$

Whence,

$$x = 1$$

Subtracting (8) from (1),

$$2y = 2$$

Whence,

$$y = 1$$

5. Given

$$\begin{cases} x^{-2} + y^{-2} = \frac{5}{3} & (1) \\ x^{-1} + y^{-1} = 1 & (2) \end{cases}$$

Squaring (2),

$$x^{-2} + 2x^{-1}y^{-1} + y^{-2} = 1 \quad (3)$$

Subtracting (1) from (3),

$$2x^{-1}y^{-1} = \frac{2}{3} \quad (4)$$

Subtracting (4) from (1),

$$x^{-2} - 2x^{-1}y^{-1} + y^{-2} = \frac{1}{3} \quad (5)$$

Evolving,

$$x^{-1} - y^{-1} = \pm \frac{1}{3} \quad (6)$$

Equation (2),

$$x^{-1} + y^{-1} = 1$$

Adding (6) and (2),

$$2x^{-1} = \frac{4}{3}, \text{ or } \frac{2}{3}$$

Dividing by 2,

$$x^{-1} = \frac{2}{3}, \text{ or } \frac{1}{3}$$

Whence,

$$x = \frac{3}{2}, \text{ or } 3$$

Subtracting (6) from (2),

$$2y^{-1} = \frac{2}{3}, \text{ or } \frac{4}{3}$$

Dividing by 2,

$$y^{-1} = \frac{1}{3}, \text{ or } \frac{2}{3}$$

Whence,

$$y = 3, \text{ or } \frac{3}{2}$$

NOTE. If it is preferred to do so, the negative exponents may be re-

moved at once, transforming (1) and (2) to $x^2 + y^2 = \frac{5x^2y^2}{9}$ and $x + y = xy$, from which $x + y = \frac{2}{3}$ and $xy = \frac{2}{3}$ are easily obtained.

$$\begin{array}{ll} 6. \text{ Given} & \begin{cases} x^3 - y^3 = 8 & (1) \\ x - y = 2 & (2) \end{cases} \end{array}$$

$$\text{Dividing (1) by (2),} \quad \frac{x^3 - y^3}{x - y} = \frac{8}{2} \quad (3)$$

$$\text{Squaring (2),} \quad x^2 - 2xy + y^2 = 4 \quad (4)$$

$$\text{Subtracting (4) from (3),} \quad 3xy = 0 \quad (5)$$

$$\text{Whence,} \quad xy = 0 \quad (6)$$

$$\text{Adding (6) to (3),} \quad x^2 + 2xy + y^2 = 4 \quad (7)$$

$$\text{Evolving,} \quad x + y = \pm 2 \quad (8)$$

$$\text{Equation (2),} \quad x - y = 2$$

$$\text{Adding (8) and (2),} \quad 2x = 4, \text{ or } 0$$

$$\text{Whence,} \quad x = 2, \text{ or } 0$$

$$\text{Subtracting (2) from (8),} \quad 2y = 0, \text{ or } -4$$

$$\text{Whence,} \quad y = 0, \text{ or } -2$$

$$\begin{array}{ll} 7. \text{ Given} & \begin{cases} x^4 + y^4 = 82 & (1) \\ xy = 3 & (2) \end{cases} \end{array}$$

$$\text{Squaring (2),} \quad x^2y^2 = 9 \quad (3)$$

$$\text{Multiplying (3) by (2),} \quad 2x^2y^2 = 18 \quad (4)$$

$$\text{Adding (4) to (1),} \quad x^4 + 2x^2y^2 + y^4 = 100 \quad (5)$$

$$\text{Subtracting (4) from (1),} \quad x^4 - 2x^2y^2 + y^4 = 64 \quad (6)$$

$$\text{Extracting square root of (5),} \quad x^2 + y^2 = \pm 10 \quad (7)$$

$$\text{Extracting square root of (6),} \quad x^2 - y^2 = \pm 8 \quad (8)$$

$$\text{Adding (7) and (8),} \quad 2x^2 = \pm 18, \text{ or } \pm 2$$

$$\text{Dividing by 2,} \quad x^2 = \pm 9, \text{ or } \pm 1$$

$$\text{Evolving,} \quad x = \pm 3, \text{ or } \pm 1, \text{ or } \pm 3\sqrt{-1}, \text{ or } \pm \sqrt{-1}$$

$$\text{Subtracting (8) from (7),} \quad 2y^2 = \pm 2, \text{ or } \pm 18$$

$$\text{Dividing by 2,} \quad y^2 = \pm 1, \text{ or } \pm 9$$

$$\text{Evolving,} \quad y = \pm 1, \text{ or } \pm 3, \text{ or } \pm \sqrt{-1}, \text{ or } \pm 3\sqrt{-1}$$

NOTE. The same results may be obtained from (2) and (7), as in Ex. 2, Art. 285, without the use of (6) and (8).

$$8. \text{ Given } \begin{cases} x - y = 8 (\sqrt{x} - \sqrt{y}) & (1) \\ \sqrt{xy} = 15 & (2) \end{cases}$$

$$\text{Dividing (1) by } \sqrt{x} - \sqrt{y}, \quad \sqrt{x} + \sqrt{y} = 8 \quad (3)$$

$$\text{Squaring (3),} \quad x + 2\sqrt{xy} + y = 64 \quad (4)$$

$$\text{Substituting (2) in (4),} \quad x + y = 34 \quad (5)$$

$$\text{Squaring (2),} \quad xy = 225 \quad (6)$$

$$\text{Squaring (5),} \quad x^2 + 2xy + y^2 = 1156 \quad (7)$$

$$\text{Multiplying (6) by 4,} \quad 4xy = 900 \quad (8)$$

$$\text{Subt. (8) from (7),} \quad x^2 - 2xy + y^2 = 256 \quad (9)$$

$$\text{Evolving,} \quad x - y = \pm 16 \quad (10)$$

$$\text{Equation (5),} \quad x + y = 34$$

$$\text{Adding (10) and (5),} \quad 2x = 50, \text{ or } 18$$

$$\text{Whence,} \quad x = 25, \text{ or } 9$$

$$\text{Subtracting (10) from (5),} \quad 2y = 18, \text{ or } 50$$

$$\text{Whence,} \quad y = 9, \text{ or } 25$$

NOTE. After obtaining equation (4), we might subtract from it four times equation (2), and the square root of the result would be $\sqrt{x} - \sqrt{y} = \pm 2$. This, combined with (3), gives $\sqrt{x} = 5$, or 3, and $\sqrt{y} = 3$, or 5, whence $x = 25$, or 9, and $y = 9$, or 25, as above.

Since we divide (1) by $\sqrt{x} - \sqrt{y}$, it is evident that $\sqrt{x} - \sqrt{y} = 0$, or $x = y$, will satisfy that equation. This, combined with (2), gives the additional roots of the equations, $x = 15$, $y = 15$. The same results may be obtained by eliminating one of the unknown quantities from (1) and (2) by substitution, without dividing by $\sqrt{x} - \sqrt{y}$. The resulting equation may be put in the form

$$x^2 - 8x^{\frac{3}{2}} + 16x = 16x - 120x^{\frac{1}{2}} + (15)^2$$

$$\text{which gives} \quad x - 4x^{\frac{1}{2}} = \pm (4x^{\frac{1}{2}} - 15)$$

$$\text{whence} \quad x = 25, \text{ or } 9, \text{ or } 15$$

$$\text{Also} \quad y = 9, \text{ or } 25, \text{ or } 15$$

The forms of the roots show that equation (1) is *symmetrical* in signs, as well as in other respects. (Art. 285, Ex. 1.) This may also be inferred from the fact that if all its signs be changed, the signs of x and y will be reversed. Equation (1) is also changed to the symmetrical form $x - 8\sqrt{x} = y - 8\sqrt{y}$ by simply transposing.

9. Given

$$\begin{cases} x + y = 72 & (1) \\ x^{\frac{1}{2}} + y^{\frac{1}{2}} = 6 & (2) \end{cases}$$

Substituting v for $x^{\frac{1}{2}}$ and z for $y^{\frac{1}{2}}$, $\begin{cases} v^2 + z^2 = 72 & (3) \\ v + z = 6 & (4) \end{cases}$

Dividing (3) by (4), $v^2 - vz + z^2 = 12$ (5)

Squaring (4), $v^2 + 2vz + z^2 = 36$ (6)

Subtracting (5) from (6), $3vz = 24$ (7)

Whence, $vz = 8$ (8)

Subtracting (8) from (5), $v^2 - 2vz + z^2 = 4$ (9)

Evolving, $v - z = \pm 2$ (10)

Equation (4), $v + z = 6$

Adding (10) and (4), $2v = 8$, or 4

Dividing by 2, $v = 4$, or 2

Whence, $x = v^2 = 16$, or 4

Subtracting (10) from (4), $2z = 4$, or 8

Dividing by 2, $z = 2$, or 4

Whence, $y = z^2 = 4$, or 16

NOTE. The value of $x^{\frac{1}{2}} y^{\frac{1}{2}}$ may be found, just as equation (8) is obtained, without the use of v and z . The remaining portion of the solution may be made to correspond exactly with the above, x and y with fractional exponents being used in place of v and z ; but, if it is preferred, $x^{\frac{1}{2}} y^{\frac{1}{2}} = 8$ may be cubed, and we then have $x + y = 72$ and $xy = 512$, from which x and y can be obtained without any further use of fractional exponents. In the latter case we must work with larger numbers, but we avoid fractional exponents; in the former we have the advantage of smaller numbers, but we must work with fractional exponents.

One of the unknown quantities may be eliminated by substitution in either of the preceding examples, and the resulting equation will be a quadratic, or in the quadratic form, except in the first example.

10. Given

$$\begin{cases} x + y : x - y :: 13 : 5 & (1) \\ y^2 + x = 25 & (2) \end{cases}$$

From (1), $5x + 5y = 13x - 13y$ (3)

Or, $18y = 8x$ (4)

Whence, $x = \frac{9y}{4}$ (5)

Substituting in (2), $y^2 + \frac{9y}{4} = 25$ (6)

Completing the square, $y^2 + \frac{9y}{4} + \frac{81}{64} = \frac{1681}{64}$ (7)

Evolving, $y + \frac{9}{8} = \pm \frac{41}{8}$

Whence, $y = 4$, or $-6\frac{1}{2}$

Substituting in (5), $x = \frac{9 \times 4}{4}$, or $-\frac{25}{4} \times \frac{9}{4}$

Reducing, $x = 9$, or $-14\frac{1}{8}$

NOTE. As these equations are not symmetrical, the only proper course is to eliminate one of the unknown quantities at once, and solve the quadratic equation thus produced according to the principles of Case I.

11. Given $\begin{cases} x + 4y = 14 & (1) \\ 4x - 2y + y^2 = 11 & (2) \end{cases}$

From (1), $x = 14 - 4y$ (3)

Subst. in (2), $56 - 16y - 2y + y^2 = 11$ (4)

Or, $y^2 - 18y = -45$ (5)

Completing square, $y^2 - 18y + 81 = 36$ (6)

Evolving, $y - 9 = \pm 6$

Whence, $y = 15$, or 3

Substituting in (3), $x = 14 - 60$, or $14 - 12$

Uniting, $x = -46$, or 2

12. Given $\begin{cases} x + 3y = 7 & (1) \\ x^2 - 3xy + 3y^2 = 7 & (2) \end{cases}$

From (1), $x = 7 - 3y$ (3)

Subs. in (2), $49 - 42y + 9y^2 - 21y + 9y^2 + 3y^2 = 7$ (4)

Whence, $21y^2 - 63y = -42$ (5)

Dividing by 21, $y^2 - 3y = -2$ (6)

Completing the square, $y^2 - 3y + \frac{9}{4} = \frac{1}{4}$ (7)

Evolving, $y - \frac{3}{2} = \pm \frac{1}{2}$

Whence, $y = 2$, or 1

Substituting in (3), $x = 7 - 6$, or $7 - 3$

Uniting, $x = 1$, or 4

13. Given	$\begin{cases} x^2 - 4y^2 = 9 & (1) \\ xy + 2y^2 = 18 & (2) \end{cases}$
Multiplying (2) by 4,	$4xy + 8y^2 = 72 \quad (3)$
Adding (1) and (3),	$x^2 + 4xy + 4y^2 = 81 \quad (4)$
Evolving,	$x + 2y = \pm 9 \quad (5)$
Dividing (1) by (5),	$x - 2y = \pm 1 \quad (6)$
Adding (5) and (6),	$2x = \pm 10$
Whence,	$x = \pm 5$
Subtracting (6) from (5),	$4y = \pm 8$
Whence,	$y = \pm 2$

NOTE. As both of the above equations are homogeneous and of the second degree, the example comes under Case II. The roots show, however, that the two equations must produce a pure quadratic by elimination. If the value of x in (2) be substituted in (1), we obtain

$$\frac{(18)^2 - 72y^2 + 4y^4}{y^2} - 4y^2 = 9$$

which reduces to

$$81y^2 = (18)^2$$

The peculiarity of this example is that the factor $x + 2y$ is common to both equations. If we divide one of these equations by the other, this factor may be canceled, and the resulting simple equation is reducible to $2x = 5y$. The example now becomes of the same nature as those of Art. 265.

14. Given	$\begin{cases} x^2 + xy + 2y^2 = 74 & (1) \\ 2x^2 + 2xy + y^2 = 73 & (2) \end{cases}$
Mult. (1) by 2,	$2x^2 + 2xy + 4y^2 = 148 \quad (3)$
Subtracting (2) from (3),	$3y^2 = 75 \quad (4)$
Dividing by 3,	$y^2 = 25 \quad (5)$
Evolving,	$y = \pm 5 \quad (6)$
Substituting in (1),	$x^2 \pm 5x + 50 = 74 \quad (7)$
Whence,	$x^2 \pm 5x = 24 \quad (8)$
Completing square,	$x^2 \pm 5x + \frac{25}{4} = \frac{121}{4} \quad (9)$
Evolving,	$x \pm \frac{5}{2} = \pm \frac{11}{2}$
Transposing,	$x = \pm \frac{11}{2} \mp \frac{5}{2}$
Uniting,	$x = \pm 3, \text{ or } \mp 8$

NOTE. The signs of the values of x are so placed that the upper ones

may correspond with the positive value of y , and the lower ones with its negative value; for if $y = +5$, $x = 3$, or -8 , and if $y = -5$, $x = 8$, or -3 .

It is evident that this example, as well as the one immediately preceding and the one immediately following it, comes under Case II. If we assume $x = vy$, the value of v will be found to be $\frac{2}{3}$, or $-\frac{2}{3}$. The solution given above is, however, greatly to be preferred.

$$\begin{array}{ll} 15. \text{ Given} & \begin{cases} x^2 + 2y^2 = 9 & (1) \\ xy + 2x^2 = 4 & (2) \end{cases} \end{array}$$

$$\text{Let} \quad y = vx \quad (3)$$

$$\text{Substituting in (1),} \quad x^2 + 2v^2x^2 = 9 \quad (4)$$

$$\text{Substituting in (2),} \quad vx^2 + 2x^2 = 4 \quad (5)$$

$$\text{From (4),} \quad x^2 = \frac{9}{1 + 2v^2} \quad (6)$$

$$\text{From (5),} \quad x^2 = \frac{4}{v + 2} \quad (7)$$

$$\text{Hence,} \quad \frac{4}{v + 2} = \frac{9}{1 + 2v^2} \quad (8)$$

$$\text{Clearing of fractions,} \quad 4 + 8v^2 = 9v + 18 \quad (9)$$

$$\text{Transposing and uniting,} \quad 8v^2 - 9v = 14 \quad (10)$$

$$\text{Multiplying by 2,} \quad 16v^2 - 18v = 28 \quad (11)$$

$$\text{Adding } \left(\frac{18}{2\sqrt{16}}\right)^2, \quad 16v^2 - 18v + \frac{81}{16} = \frac{529}{16} \quad (12)$$

$$\text{Evolving,} \quad 4v - \frac{9}{4} = \pm \frac{23}{4} \quad (13)$$

$$\text{Whence,} \quad 4v = 8, \text{ or } -\frac{7}{2}$$

$$\text{Or,} \quad v = 2, \text{ or } -\frac{7}{8}$$

$$\text{Substituting in (7),} \quad x^2 = \frac{4}{2 + 2}, \text{ or } \frac{4}{-\frac{7}{8} + 2}$$

$$\text{Reducing,} \quad x^2 = 1, \text{ or } \frac{32}{9}$$

$$\text{Evolving,} \quad x = \pm 1, \text{ or } \pm \frac{4}{3}\sqrt{2}$$

$$\text{Substituting in (3),} \quad y = \pm 1 \times 2, \text{ or } \pm \frac{4}{3}\sqrt{2} \times \left(-\frac{7}{8}\right)$$

$$\text{Reducing,} \quad y = \pm 2, \text{ or } \mp \frac{7}{6}\sqrt{2}$$

NOTE. The above equations may also be solved by substituting in (1) the value of y found in (2). See Art. 284.

$$16. \text{ Given } \begin{cases} x^2 + y^2 + x + y = 18 & (1) \\ 2xy = 12 & (2) \end{cases}$$

$$\text{Adding (2) to (1), } (x+y)^2 + (x+y) = 30 \quad (3)$$

$$\text{Compl. square, } (x+y)^2 + (x+y) + \frac{1}{4} = 30\frac{1}{4} = 12\frac{1}{4} \quad (4)$$

$$\text{Evolving, } x + y + \frac{1}{2} = \pm \frac{1}{2} \quad (5)$$

$$\text{Whence, } x + y = 5, \text{ or } -6 \quad (6)$$

$$\text{Subtracting (6) from (1), } x^2 + y^2 = 13, \text{ or } 24 \quad (7)$$

$$\text{Subt. (2) from (7), } x^2 - 2xy + y^2 = 1, \text{ or } 12 \quad (8)$$

$$\text{Evolving, } x - y = \pm 1, \text{ or } \pm 2\sqrt{3} \quad (9)$$

$$\text{Equation (6), } x + y = 5, \text{ or } -6$$

$$\text{Adding (9) and (6), } 2x = 6, \text{ or } 4, \text{ or } -6 \pm 2\sqrt{3}$$

$$\text{Whence, } x = 3, \text{ or } 2, \text{ or } -3 \pm \sqrt{3}$$

$$\text{Subtracting (9) from (6), } 2y = 4, \text{ or } 6, \text{ or } -6 \mp 2\sqrt{3}$$

$$\text{Whence, } y = 2, \text{ or } 3, \text{ or } -3 \mp \sqrt{3}$$

NOTE. Equation (8) may also be obtained from (2) and (6) in the usual way, without the use of (1).

THEORY OF QUADRATIC EQUATIONS.

(ART. 290, p. 272.)

$$\begin{aligned} 1. \text{ Given } & x^2 - 4x + 3 = 0 \\ \text{Transp. and compl. square, } & x^2 - 4x + 4 = 1 \\ \text{Evolving, } & x - 2 = \pm 1 \\ \text{Whence, } & x = 3, \text{ or } 1 \\ \text{Hence, } & x^2 - 4x + 3 = (x - 3)(x - 1) = 0 \end{aligned}$$

$$\begin{aligned} 2. \text{ Given } & x^2 - \frac{x}{6} - \frac{1}{3} = 0 \\ \text{Transp. and compl. square, } & x^2 - \frac{x}{6} + \frac{1}{144} = \frac{149}{144} \\ \text{Evolving, } & x - \frac{1}{12} = \pm \frac{1}{12} \\ \text{Whence, } & x = \frac{2}{3}, \text{ or } -\frac{1}{3} \\ \text{Hence, } & x^2 - \frac{x}{6} - \frac{1}{3} = (x - \frac{2}{3})(x + \frac{1}{3}) = 0 \end{aligned}$$

3. Given $x^2 - 7x + 12 = 0$
 Transp. and compl. square, $x^2 - 7x + \frac{49}{4} = \frac{1}{4}$
 Evolving, $x - \frac{7}{2} = \pm \frac{1}{2}$
 Whence, $x = 4$, or 3
 Ans. $(x - 4)(x - 3) = 0$

4. Given $x^2 + 6x + 8 = 0$
 Transp. and compl. square, $x^2 + 6x + 9 = -1$
 Evolving, $x + 3 = \pm 1$
 Whence, $x = -2$, or -4
 Ans. $(x + 2)(x + 4) = 0$

NOTE. The principles established in Art. 289 will determine whether the roots obtained for any quadratic equation are correct; and in some cases they will enable us to obtain those roots by inspection. Thus, in the third example, it is evident that 4 and 3 are the only numbers whose sum is 7, and whose product is 12.

(ART. 291, pp. 272, 273.)

1. Assume $x^2 - 5x + 6 = 0$. It is evident that 3 and 2 are the only numbers whose sum is 5 and whose product is 6; hence they are the roots of the equation, and
 $x^2 - 5x + 6 = (x - 3)(x - 2)$.

2. Assume $x^2 - \frac{x}{12} - \frac{5}{8} = 0$
 Completing square, $x^2 - \frac{x}{12} + \frac{1}{144} = \frac{361}{144}$
 Evolving, $x - \frac{1}{24} = \pm \frac{19}{12}$
 Whence, $x = \frac{5}{6}$, or $-\frac{3}{4}$
 Therefore, $x^2 - \frac{x}{12} - \frac{5}{8} = (x - \frac{5}{6})(x + \frac{3}{4})$

3. Assume, $x^2 + 3x - 28 = 0$
 Completing square, $x^2 + 3x + \frac{9}{4} = \frac{121}{4}$
 Evolving, $x + \frac{3}{2} = \pm \frac{11}{2}$
 Whence, $x = 4$, or -7
 $x^2 + 3x - 28 = (x - 4)(x + 7)$, Ans.

4. Assume $z^2 + 18z + 80 = 0$. It is evident that -10 and -8 are the only numbers whose sum is -18 , and whose product is 80 ; hence they are the roots of the equation, and

$$z^2 + 18z + 80 = (z + 10)(z + 8), \text{ Ans.}$$

FORMATION OF EQUATIONS.

(ART. 292, pp. 273, 274.)

$$2. (x - 4)(x - 5) = x^2 - 9x + 20 = 0$$

$$\text{Or, } x^2 - 9x = -20$$

$$3. (x - 6)(x - 7) = x^2 - 13x + 42 = 0$$

$$\text{Or, } x^2 - 13x = -42, \text{ Ans.}$$

$$4. (x + 1)(x + 2) = x^2 + 3x + 2 = 0$$

$$\text{Or, } x^2 + 3x = -2$$

$$5. (x - 20)(x + 30) = x^2 + 10x - 600 = 0$$

$$\text{Or, } x^2 + 10x = 600$$

$$6. (x - \frac{2}{5})(x + \frac{14}{5}) = x^2 + \frac{12}{5}x - \frac{28}{5} = 0$$

$$\text{Or, } x^2 + \frac{12}{5}x = \frac{28}{5}$$

$$\text{Mult. by } 10, 10x^2 + 12x = 28$$

$$7. (x - a)(x - b) = x^2 - ax - bx + ab = 0$$

$$\text{Or, } x^2 - (a + b)x = -ab$$

$$8. (x - \overline{m + n})(x - \overline{m - n}) = x^2 - 2mx + m^2 - n^2 = 0$$

$$\text{Or, } x^2 - 2mx = n^2 - m^2$$

NOTE. The principles of Art. 289 will enable us to write out at once an equation having any given roots. Thus the coefficient of x in the last equation will be $-2m$ because $m + n + m - n = 2m$, and the second member is $-(m^2 - n^2)$ because $(m + n)(m - n) = m^2 - n^2$.

PROBLEMS

LEADING TO AFFECTED QUADRATIC EQUATIONS.

(ART. 299, pp. 277-285.)

2. Let $x = \text{the number.}$

$$\text{Then,} \quad x - \left(\frac{x}{4}\right)^2 = 3$$

$$\text{Or,} \quad x^2 - 16x = -48$$

$$\text{Completing square, } x^2 - 16x + 64 = 16$$

$$\text{Evolving,} \quad x - 8 = \pm 4$$

$$\text{Whence,} \quad x = 12, \text{ or } 4$$

$$\text{VERIFICATION.} \quad \begin{cases} 12 - 3^2 = 3 \\ 4 - 1^2 = 3 \end{cases}$$

5. Let $x = \text{the cost.}$

$$\text{Then} \quad x = \text{the gain per cent.,}$$

$$\text{and} \quad \frac{x}{100} \times x = \frac{x^2}{100} = \text{the whole gain.}$$

$$\text{Therefore,} \quad x + \frac{x^2}{100} = 56$$

$$\text{Or,} \quad x^2 + 100x = 5600$$

$$\text{Completing square, } x^2 + 100x + 2500 = 8100$$

$$\text{Evolving,} \quad x + 50 = \pm 90$$

$$\text{Whence,} \quad x = 40, \text{ or } -140$$

6. Let $x = \text{the number of chickens.}$

$$\text{Then} \quad \frac{96}{x} = \text{cost of one chicken,}$$

$$\text{and} \quad \frac{27x}{2} = \text{am't rec'd for the chickens.}$$

$$\text{Therefore,} \quad \frac{27x}{2} - 96 = \frac{96}{x}$$

$$\text{Clearing of fractions,} \quad 27x^2 - 192x = 192$$

$$\text{Dividing by 27,} \quad x^2 - \frac{64}{9}x = \frac{64}{9}$$

$$\text{Completing square, } x^2 - \frac{64}{9}x + \left(\frac{32}{9}\right)^2 = \frac{1600}{81}$$

$$\text{Evolving,} \quad x - \frac{32}{9} = \pm \frac{40}{9}$$

$$\text{Whence,} \quad x = 8, \text{ or } -\frac{8}{9}$$

7. Let $x =$ the number of pages.
 Then $x - 5 =$ estimated no. pages,
 $\frac{80}{x - 5} =$ estimated cost per page,
 and $\frac{67.50}{x} = \frac{135}{2x} =$ am't received per page.

Therefore,
$$\frac{80}{x - 5} - \frac{135}{2x} = \frac{1}{2}$$

Clearing of fractions, $160x - 135x + 675 = x^2 - 5x$

Whence, $x^2 - 30x = 675$

Completing square, $x^2 - 30x + 225 = 900$

Evolving, $x - 15 = \pm 30$

Whence, $x = 45$, or -15

If "5 pages more" be changed to "5 pages less," and "abatement" to "*additional charge*," then 15 will be the true number of pages.

8. Let $x =$ no. persons at first,
 and $x - 2 =$ no. persons who paid.
 Then $\frac{8.75}{x} = \frac{35}{4x} =$ am't each should have paid,
 and $\frac{8.75}{x - 2} = \frac{35}{4x - 8} =$ am't each did pay.

Therefore,
$$\frac{35}{4x - 8} - \frac{35}{4x} = \frac{1}{2}$$

Clearing of fractions, $140x - 140x + 280 = 8x^2 - 16x$

Whence, $8x^2 - 16x = 280$

Dividing by 8, $x^2 - 2x = 35$

Completing square, $x^2 - 2x + 1 = 36$

Evolving, $x - 1 = \pm 6$

Whence, $x = 7$, or -5

The number 5 is the answer to a problem like the above, except that two *additional* persons came, and each of the company had therefore 50 cts. *less* to pay.

9. Let $x =$ original no. persons,
 and $x + 2 =$ whole no. persons.

$$\begin{array}{ll} \text{Then} & \frac{1000}{x} = \text{sum each expected originally,} \\ \text{and} & \frac{1000}{x+2} = \text{sum each will receive.} \end{array}$$

Therefore, $\frac{1000}{x} - \frac{1000}{x+2} = 25$

Clear. of fractions, $1000x + 2000 - 1000x = 25x^2 + 50x$

Or, $25x^2 + 50x = 2000$

Dividing by 25, $x^2 + 2x = 80$

Completing square, $x^2 + 2x + 1 = 81$

Evolving, $x + 1 = \pm 9$

Whence, $x = 8, \text{ or } -10$

The number 10 is the answer to a problem similar to the one given, except that two claimants *disappear*, and each receives \$25 *more* than he expected.

10. Let $x = \text{width of frame.}$

Then $18 + 2x = \text{outside length of frame,}$

and $12 + 2x = \text{outside width of frame.}$

Also $18 \times 12 = 216 = \text{surface of glass,}$

and $(18 + 2x)(12 + 2x) = \text{whole surface.}$

Therefore, $(18 + 2x)(12 + 2x) = 2 \times 216$

Expanding, $216 + 60x + 4x^2 = 432$

Or, $4x^2 + 60x = 216$

Dividing by 4, $x^2 + 15x = 54$

Completing square, $x^2 + 15x + \frac{225}{4} = \frac{441}{4}$

Evolving, $x + \frac{15}{2} = \pm \frac{21}{2}$

Whence, $x = 3, \text{ or } -18$

NOTE. The equation $4x^2 + 60x = 216$ may be obtained at once by estimating the surface of the frame. $4x^2$ is the area of the four corners; there are two sides each 18 by x , and two ends each 12 by x ; hence the whole surface of the frame is $4x^2 + 60x$, which, according to the conditions of the problem, is equal to the surface of the glass.

12. FIRST SOLUTION.

Let $x = \text{one part,}$

and $y = \text{the other part.}$

Then,	$x + y = 40$	(1)
and	$x^2 + y^2 = 818$	(2)
Squaring (1),	$x^2 + 2xy + y^2 = 1600$	(3)
Subtracting (2) from (3),	$2xy = 782$	(4)
Subtr. (4) from (2),	$x^2 - 2xy + y^2 = 36$	(5)
Evolving,	$x - y = \pm 6$	(6)
Equation (1),	$x + y = 40$	
Adding (6) and (1),	$2x = 46, \text{ or } 34$	
Whence,	$x = 23, \text{ or } 17$	
Subtracting (6) from (1),	$2y = 34, \text{ or } 46$	
Whence,	$y = 17, \text{ or } 23$	

NOTE. Either letter may also be eliminated by substitution, or only one unknown quantity may be used, if preferred.

SECOND SOLUTION.

Let	$x - y = \text{one part,}$	
and	$x + y = \text{the other part.}$	
Their sum,	$2x = 40$	(1)
The sum of their squares,	$2x^2 + 2y^2 = 818$	(2)
From (1),	$x = 20$	(3)
Squaring,	$x^2 = 400$	(4)
From (2),	$x^2 + y^2 = 409$	(5)
Subtracting (4) from (5),	$y^2 = 9$	(6)
Evolving,	$y = \pm 3$	(7)
From (3) and (7),	$x - y = 17, \text{ or } 23$	
Also,	$x + y = 23, \text{ or } 17$	

13. Let	$x = \text{one part,}$	
and	$y = \text{the other part.}$	
Then,	$x + y = 60$	(1)
and	$xy : x^2 + y^2 :: 2 : 5$	(2)
From (2),	$2x^2 + 2y^2 = 5xy$	(3)
Or,	$x^2 - \frac{5}{2}xy + y^2 = 0$	(4)
Squaring (1),	$x^2 + 2xy + y^2 = 3600$	(5)

$$\begin{array}{ll}
\text{Subtracting (4) from (5),} & \frac{2}{3}xy = 3600 \quad (6) \\
\text{Whence,} & xy = 800 \quad (7) \\
\text{Or,} & 4xy = 3200 \quad (8) \\
\text{Subtracting (8) from (5), } x^2 - 2xy + y^2 = 400 & (9) \\
\text{Evolving,} & x - y = \pm 20 \quad (10) \\
\text{Equation (1),} & x + y = 60 \\
\hline
\text{Adding (10) and (1),} & 2x = 80, \text{ or } 40 \\
\text{Whence,} & x = 40, \text{ or } 20 \\
\text{Subtracting (10) from (1),} & 2y = 40, \text{ or } 80 \\
\text{Whence,} & y = 20, \text{ or } 40
\end{array}$$

NOTE. This problem may be solved by the various methods indicated in connection with the solutions of the last example. If $x - y$ and $x + y$ are used, the equations become $2x = 60$ and $x^2 - y^2 : 2x^2 + 2y^2 :: 2 : 5$, or $x = 30$ and $x^2 = 9y^2$, from which $y = \pm 10$, and the numbers are 20 and 40, as before.

$$\begin{array}{ll}
15. \text{ Let} & x = \text{cost of 1st per yard, in doll.,} \\
\text{and} & y = \text{cost of 2d per yard, in doll.} \\
\text{Then} & 100y = \text{gain per cent of first,} \\
\text{and} & 100y = \text{loss per cent of second.}
\end{array}$$

$$\text{Also } x \times \frac{100y}{100} = xy = \text{gain on a yard of first,}$$

$$\text{and } y \times \frac{100y}{100} = y^2 = \text{loss on a yard of second.}$$

$$\text{Therefore,} \quad 80x + 80y = 60 \quad (1)$$

$$\text{Also,} \quad 80xy - 80y^2 = 5 \quad (2)$$

$$\text{From (1),} \quad 4x + 4y = 3 \quad (3)$$

$$\text{From (2),} \quad 16xy - 16y^2 = 1 \quad (4)$$

$$\text{Mult. (3) by } 4y, \quad 16xy + 16y^2 = 12y \quad (5)$$

$$\text{Subt. (4) from (5),} \quad 32y^2 = 12y - 1 \quad (6)$$

$$\text{Whence,} \quad y^2 - \frac{3}{8}y = -\frac{1}{32} \quad (7)$$

$$\text{Completing square, } y^2 - \frac{3}{8}y + \frac{9}{64} = \frac{9}{64} - \frac{1}{32} \quad (8)$$

$$\text{Evolving,} \quad y - \frac{3}{16} = \pm \frac{1}{16} \quad (9)$$

$$\text{Whence,} \quad y = \frac{1}{4}, \text{ or } \frac{1}{2}$$

From (3),
$$x = \frac{3-4y}{4} = \frac{3-1}{4}, \text{ or } \frac{3-\frac{1}{2}}{4}$$

Reducing,
$$x = \frac{1}{2}, \text{ or } \frac{5}{8}$$

It is evident that $\$ \frac{1}{2} = 50$ cts., $\$ \frac{5}{8} = 62\frac{1}{2}$ cts., $\$ \frac{1}{4} = 25$ cts., and $\$ \frac{1}{8} = 12\frac{1}{2}$ cts.

NOTE. If x and y represent the cost of each in *cents*, then y will represent the gain and loss per cent, and the equations will become

$$80x + 80y = 6000, \quad \text{and} \quad \frac{80xy}{100} - \frac{80y^2}{100} = 500,$$

whence
$$x + y = 75, \quad \text{and} \quad xy - y^2 = 625.$$

These give $2y^2 - 75y = -625$, $y = 25$, or $12\frac{1}{2}$, and $x = 50$, or $62\frac{1}{2}$.

16. Let
$$\begin{array}{l} x = \text{the greater,} \\ \text{and} \quad y = \text{the less.} \end{array}$$

Then,
$$x(x+y) = 144 \quad (1)$$

and
$$y(x-y) = 14 \quad (2)$$

Expanding (1),
$$x^2 + xy = 144 \quad (3)$$

Expanding (2),
$$xy - y^2 = 14 \quad (4)$$

From (4),
$$x = \frac{14 + y^2}{y} \quad (5)$$

Subst. in (3),
$$\frac{196 + 28y^2 + y^4}{y^2} + 14 + y^2 = 144 \quad (6)$$

Cl. of fract.,
$$196 + 28y^2 + y^4 + 14y^2 + y^4 = 144y^2 \quad (7)$$

Whence,
$$2y^4 - 102y^2 = -196 \quad (8)$$

Multiplying by 2,
$$4y^4 - 204y^2 = -392 \quad (9)$$

Adding $(192)^2$,
$$4y^4 - 204y^2 + 2601 = 2209 \quad (10)$$

Evolving,
$$2y^2 - 51 = \pm 47 \quad (11)$$

Whence,
$$2y^2 = 98, \text{ or } 4$$

Dividing by 2,
$$y^2 = 49, \text{ or } 2$$

Evolving,
$$y = \pm 7, \text{ or } \pm \sqrt{2}$$

Substituting in (5),
$$x = \frac{14 + 49}{\pm 7}, \text{ or } \frac{14 + 2}{\pm \sqrt{2}}$$

Reducing,
$$x = \pm 9, \text{ or } \pm 8\sqrt{2}$$

NOTE. As both the equations are homogeneous and of the second degree, they may be solved by assuming $y = vx$, according to the method of Art. 284.

17. Let $x =$ no. bushels.

Then $\frac{60}{x} =$ price paid per bush., in dollars,

and $\frac{54}{x-15} =$ price rec'd per bush., in dollars.

Therefore,

$$\frac{54}{x-15} - \frac{60}{x} = \frac{1}{10}$$

Clearing of fractions, $540x - 600x + 9000 = x^2 - 15x$

Whence,

$$x^2 + 45x = 9000$$

Completing square,

$$x^2 + 45x + \left(\frac{45}{2}\right)^2 = \frac{32025}{4}$$

Evolving,

$$x + \frac{45}{2} = \pm \frac{185}{2}$$

Whence,

$$x = 75, \text{ or } -120$$

If he had *added* 15 bushels, instead of reserving 15, and *lost* 10 cents a bushel by selling for \$54, then 120 bushels would be the true answer.

18. Let

$x =$ digit in tens' place,

and

$y =$ digit in units' place.

Then

$$10x + y = \text{the number.}$$

From the conditions,

$$x + y = 15 \quad (1)$$

Also,

$$xy + 31 = 10y + x \quad (2)$$

From (1),

$$y = 15 - x \quad (3)$$

Substituting in (2), $15x - x^2 + 31 = 150 - 10x + x \quad (4)$

Or,

$$x^2 - 24x = -119 \quad (5)$$

Completing square, $x^2 - 24x + 144 = 25 \quad (6)$

Evolving,

$$x - 12 = \pm 5$$

Whence,

$$x = 17, \text{ or } 7$$

Substituting in (3),

$$y = 15 - 17, \text{ or } 15 - 7$$

Uniting,

$$y = -2, \text{ or } 8$$

Hence,

$$10x + y = 78$$

19. Let

$x =$ no. gold coins,

and

$y =$ no. silver coins.

Then

$xy =$ value of gold coins, in dollars,

and

$xy =$ value of silver coins, in cents.

From the conditions, $x + y = 8$ (1)

Also, $100xy + xy = 1515$ (2)

From (2), $xy = 15$ (3)

Or, $y = \frac{15}{x}$ (4)

Substituting in (1), $x + \frac{15}{x} = 8$ (5)

Whence, $x^2 - 8x = -15$ (6)

Completing square, $x^2 - 8x + 16 = 1$ (7)

Evolving, $x - 4 = \pm 1$ (8)

Whence, $x = 5$, or 3

Substituting in (4), $y = \frac{15}{5}$, or $\frac{15}{3}$

Or, $y = 3$, or 5

NOTE. The above equations being symmetrical, they can of course be solved by the methods of Art. 285, as we have done many times where the sum and product have been given.

The problem can also be readily solved by the use of one unknown quantity.

20. Let $x = \text{price per dozen.}$

Then $\frac{x}{12} = \text{price of each egg,}$

$$12 \div \frac{x}{12} = \frac{144}{x} = \text{no. eggs for 12 cts.,}$$

$$12 \div \left(\frac{144}{x} + 2 \right) = \frac{6x}{72 + x} = \text{new price per egg,}$$

$$\text{and } \frac{6x}{72 + x} \times 12 = \frac{72x}{72 + x} = \text{new price per dozen.}$$

Therefore, $x - 1 = \frac{72x}{72 + x}$

Clearing of fractions, $72x + x^2 - 72 - x = 72x$

Whence, $x^2 - x = 72$

Completing square, $x^2 - x + \frac{1}{4} = 72\frac{1}{4} = 28\frac{9}{4}$

Evolving, $x - \frac{1}{2} = \pm 1\frac{7}{2}$

Whence, $x = 9$, or -8

The number 8 is the answer to the problem: What

are eggs a dozen, when two *less* for twelve cents *raises* the price one cent per dozen?

21. Let $x =$ no. panels on one side,
 and $y =$ no. panels on the other.
 Then $4x =$ length of one side, in yards,
 and $4y =$ length of the other, in yards.
 Also 1 A. 32 sq. rds. = 5808 square yards.
-
- From the conditions, $2(x + y) = 88$ (1)
 Also, $4x \times 4y = 5808$ (2)
-
- From (1), $x + y = 44$ (3)
 Squaring (3), $x^2 + 2xy + y^2 = 1936$ (4)
 From (2), $4xy = 1452$ (5)
-
- Subtracting (5) from (4), $x^2 - 2xy + y^2 = 484$ (6)
 Evolving, $x - y = \pm 22$ (7)
 Equation (3), $x + y = 44$
-
- Adding (7) and (3), $2x = 66$, or 22
 Whence, $x = 33$, or 11
 Subtracting (7) from (3), $2y = 22$, or 66
 Whence, $y = 11$, or 33

22. Let $x, x + 1, x + 2$, and $x + 3 =$ the numbers.
 Then, $10x + x + 1 = (x + 2)(x + 3)$
 Expanding, $10x + x + 1 = x^2 + 5x + 6$
 Whence, $x^2 - 6x = -5$
 Compl. square, $x^2 - 6x + 9 = 4$
 Evolving, $x - 3 = \pm 2$
 Whence, $x = 5$, or 1
 $x + 1 = 6$, or 2
 $x + 2 = 7$, or 3
 $x + 3 = 8$, or 4

23. Let $x =$ speed of the coach per hour,
 and $x - 5 =$ speed in returning.

$$\begin{array}{ll} \text{Then} & \frac{6}{x} = \text{no. hours going, in coach,} \\ \text{and} & \frac{6}{x-5} = \text{no. hours returning, on foot.} \end{array}$$

$$\begin{array}{ll} \text{Therefore,} & \frac{6}{x} + \frac{50}{60} = \frac{6}{x-5} \\ \text{Clearing of fractions,} & 36x - 180 + 5x^2 - 25x = 36x \\ \text{Whence,} & 5x^2 - 25x = 180 \\ \text{Dividing by 5,} & x^2 - 5x = 36 \\ \text{Completing square,} & x^2 - 5x + \frac{25}{4} = \frac{152}{4} \\ \text{Evolving,} & x - \frac{5}{2} = \pm \frac{12}{2} \\ \text{Whence,} & x = 9, \text{ or } -4 \end{array}$$

It will be seen that the second answer, -4 , is his speed in returning. It would be his speed in going if he had returned at a rate 5 miles *more* per hour, and had been 50 minutes *less* in returning than in going.

$$\begin{array}{ll} 24. \text{ Let} & x = \text{no. rec'd. by the gentleman,} \\ \text{and} & x + 2 = \text{number bought by the boy.} \\ \text{Then} & \frac{12}{x} = \text{price paid by the gentleman,} \\ \text{and} & \frac{12}{x+2} = \text{market price.} \end{array}$$

$$\begin{array}{ll} \text{Therefore,} & \frac{12}{x+2} + \frac{1}{15} = \frac{12}{x} \\ \text{Clearing of fractions,} & 180x + x^2 + 2x = 180x + 360 \\ \text{Or,} & x^2 + 2x = 360 \\ \text{Completing square,} & x^2 + 2x + 1 = 361 \\ \text{Evolving,} & x + 1 = \pm 19 \\ \text{Whence,} & x = 18, \text{ or } -20 \end{array}$$

The second root of the equation, -20 , is the number of peaches that the boy bought; or, it is the number that the gentleman would have received had the boy *added* two peaches, and thus caused the gentleman to pay at the rate of a cent for 15 *less* than the market price.

25. Let $x = A$'s speed per hour,
 and $x - \frac{1}{5} = B$'s speed per hour.
 Then $\frac{5}{x} = A$'s time for running 5 miles,
 and $\frac{5}{x - \frac{1}{5}} = \frac{25}{5x - 1} = B$'s time for running 5 miles.

Therefore,
$$\frac{5}{x} + \frac{2\frac{1}{2}}{60} = \frac{25}{5x - 1}$$

Dividing by 5,
$$\frac{1}{x} + \frac{1}{120} = \frac{5}{5x - 1}$$

Clearing of fractions, $600x - 120 + 5x^2 - x = 600x$

Whence, $5x^2 - x = 120$

Multiplying by 20, $100x^2 - 20x = 2400$

Completing square, $100x^2 - 20x + 1 = 2401$

Evolving, $10x - 1 = \pm 49$

Whence, $10x = 50$, or -48

Dividing by 10, $x = 5$, or $-4\frac{4}{5}$

Whence, $x - \frac{1}{5} = 4\frac{4}{5}$, or -5

The negative results are to be interpreted as in the last two examples.

NOTE. Either of the last three problems can be readily solved with two unknown quantities. Thus, in Prob. 25, we should have the equations $x - y = \frac{1}{5}$ and $\frac{5}{y} - \frac{5}{x} = \frac{5}{120}$, which last may be expressed $y^{-1} - x^{-1} = \frac{1}{120}$, or $x - y = \frac{xy}{120}$. This, combined with the first equation, gives the difference equal to $\frac{1}{5}$, and the product equal to 24, from which both of the values of x and y are readily obtained.

26. Let $x =$ height of the room, in feet.

Then $2x =$ breadth of the room,

$2(40 + 2x) =$ distance round the room,

$80x + 4x^2 =$ surface of the walls, in feet,

and $\frac{80x + 4x^2}{9} =$ surface of the walls, in yards.

Therefore,
$$\frac{8}{9} \left(\frac{80x + 4x^2}{9} \right) = 71\frac{1}{2}$$

$$\begin{array}{ll} \text{Or,} & \frac{20x + x^2}{6} = \frac{143}{2} \\ \text{Clearing of fractions,} & x^2 + 20x = 429 \\ \text{Completing square,} & x^2 + 20x + 100 = 529 \\ \text{Evolving,} & x + 10 = \pm 23 \\ \text{Whence,} & x = 13, \text{ or } -33 \end{array}$$

$$\begin{array}{ll} 27. \text{ Let} & x = \text{breadth of glass,} \\ \text{and} & 2x = \text{length of glass.} \\ \text{Then} & 2x^2 = \text{surface of glass,} \\ \text{and} & 6x = \text{length of frame.} \end{array}$$

$$\text{Therefore, } 2x^2 \times 1.25 - 6x \times .75 = 22.00$$

$$\begin{array}{ll} \text{Or,} & \frac{5x^2}{2} - \frac{9x}{2} = 22 \\ \text{Clearing of fractions,} & 5x^2 - 9x = 44 \\ \text{Multiplying by 20,} & 100x^2 - 180x = 880 \\ \text{Compl. square,} & 100x^2 - 180x + 81 = 961 \\ \text{Evolving,} & 10x - 9 = \pm 31 \\ \text{Whence,} & 10x = 40, \text{ or } -22 \\ \text{Dividing by 10,} & x = 4, \text{ or } -2\frac{1}{5} \\ \text{Multiplying by 2,} & 2x = 8, \text{ or } -4\frac{2}{5} \end{array}$$

If the cost of both glass and frame were \$ 22, then $2\frac{1}{5}$ and $4\frac{2}{5}$ would be the dimensions.

$$\begin{array}{ll} 28. \text{ Let} & x = \text{rate of one detachment,} \\ \text{and} & x + \frac{1}{4} = \text{rate of the other.} \\ \text{Then} & \frac{39}{x} = \text{time of one detachment,} \\ \text{and} & \frac{39}{x + \frac{1}{4}} = \frac{156}{4x + 1} = \text{time of the other.} \end{array}$$

$$\begin{array}{ll} \text{Therefore,} & \frac{156}{4x + 1} + 1 = \frac{39}{x} \\ \text{Clearing of fractions,} & 156x + 4x^2 + x = 156x + 39 \\ \text{Or,} & 4x^2 + x = 39 \\ \text{Multiplying by 16,} & 64x^2 + 16x = 624 \\ \text{Completing square,} & 64x^2 + 16x + 1 = 625 \end{array}$$

Evolving,

$$8x + 1 = \pm 25$$

Whence,

$$8x = 24, \text{ or } -26$$

Dividing by 8,

$$x = 3, \text{ or } -3\frac{1}{4}$$

Adding $\frac{1}{4}$,

$$x + \frac{1}{4} = 3\frac{1}{4}, \text{ or } -3$$

The negative roots show what would be the rates of each if the conditions were reversed. It will be seen that this belongs to the same class as Probs. 23, 24, and 25.

29. Let

 x and y represent the numbers.

Then,

$$x + y = 6 \quad (1)$$

and

$$xy = 10 \quad (2)$$

Squaring (1),

$$x^2 + 2xy + y^2 = 36 \quad (3)$$

Multiplying (2) by 4,

$$4xy = 40 \quad (4)$$

Subtracting (4) from (3), $x^2 - 2xy + y^2 = -4$ (5)

Evolving,

$$x - y = \pm 2\sqrt{-1} \quad (6)$$

Equation (1),

$$x + y = 6$$

Adding (6) and (1),

$$2x = 6 \pm 2\sqrt{-1}$$

Whence,

$$x = 3 \pm \sqrt{-1}$$

Subtracting (6) from (1),

$$2y = 6 \mp 2\sqrt{-1}$$

Whence,

$$y = 3 \mp \sqrt{-1}$$

NOTE. The same results may be obtained by eliminating one of the letters by substitution; by the use of one unknown quantity; or by the use of $x + y$ and $x - y$ to represent the numbers.

30. Let

 $x^2 =$ area of the first,

and

 $y^2 =$ area of the second.

Then

$$x^2 \times 2.25 = \frac{9x^2}{4} = \text{value of the first,}$$

and

$$y^2 \times 2.25 = \frac{9y^2}{4} = \text{value of the second.}$$

Therefore,

$$4x + 4y = 200 \quad (1)$$

and

$$x^2 + y^2 = 1300 \quad (2)$$

From (1),

$$x + y = 50 \quad (3)$$

$$\begin{array}{ll}
 \text{Squaring (3),} & x^2 + 2xy + y^2 = 2500 \quad (4) \\
 \text{Subtracting (2) from (4),} & 2xy = 1200 \quad (5) \\
 \text{Subtracting (5) from (2),} & x^2 - 2xy + y^2 = 100 \quad (6) \\
 \text{Evolving,} & x - y = \pm 10 \quad (7) \\
 \text{Equation (3),} & x + y = 50 \\
 \text{Adding (3) and (7),} & 2x = 60, \text{ or } 40 \\
 \text{Whence,} & x = 30, \text{ or } 20 \\
 \text{Squaring,} & x^2 = 900, \text{ or } 400 \\
 \text{Subtracting (7) from (3),} & 2y = 40, \text{ or } 60 \\
 \text{Whence,} & y = 20, \text{ or } 30 \\
 \text{Squaring,} & y^2 = 400, \text{ or } 900 \\
 \text{Value of the first,} & \frac{9x^2}{4} = 2025, \text{ or } 900 \\
 \text{Value of the second,} & \frac{9y^2}{4} = 900, \text{ or } 2025
 \end{array}$$

31. Let x = price of 1 lb. of tea, in dollars,
and y = price of 1 lb. of coffee, in dollars.

$$\begin{array}{ll}
 \text{Then,} & 80x + 100y = 65 \quad (1) \\
 \text{and} & \frac{20}{y} - \frac{10}{x} = 60 \quad (2) \\
 \text{From (1),} & 16x + 20y = 13 \quad (3) \\
 \text{From (2),} & 2x - y = 6xy \quad (4) \\
 \text{From (4),} & y = \frac{2x}{6x + 1} \quad (5) \\
 \text{Substituting (5) in (3),} & 16x + \frac{40x}{6x + 1} = 13 \quad (6) \\
 \text{Clearing of fract.,} & 96x^2 + 16x + 40x = 78x + 13 \quad (7) \\
 \text{Whence,} & 96x^2 - 22x = 13 \quad (8) \\
 \text{Multiplying by 6,} & 576x^2 - 132x = 78 \quad (9) \\
 \text{Adding } \left(\frac{132}{2\sqrt{576}}\right)^2, & 576x^2 - 132x + \frac{121}{16} = \frac{1369}{16} \quad (10) \\
 \text{Evolving,} & 24x - \frac{11}{4} = \pm \frac{17}{4} \quad (11) \\
 \text{Whence,} & 24x = 12, \text{ or } -\frac{13}{2} \\
 \text{Dividing by 24,} & x = \frac{1}{2}, \text{ or } -\frac{13}{48}
 \end{array}$$

Substituting in (5), $y = \frac{1}{3+1}$, or $\frac{-\frac{1}{2}}{-\frac{1}{8}+1}$

Reducing, $y = \frac{1}{4}$, or $\frac{1}{2}$

Reducing fractions of a dollar to cents, we have $x = 50$, or $-27\frac{1}{2}$, and $y = 25$, or $86\frac{3}{4}$.

32. Let x = first number of sheep,
and y = second number of sheep.

Then $\frac{80}{y}$ = price of first, per sheep,

and $\frac{180}{x}$ = price of second, per sheep.

Also $\frac{80x}{y}$ = sum received for first,

and $\frac{180y}{x}$ = sum received for second.

Then, $x + y = 100$ (1)

and $\frac{180y}{x} = \frac{80x}{y}$ (2)

Clearing (2) of fractions, $180y^2 = 80x^2$ (3)

Dividing by 20, $9y^2 = 4x^2$ (4)

Evolving, $3y = \pm 2x$ (5)

Dividing by 3, $y = \pm \frac{2x}{3}$ (6)

Substituting (6) in (1), $x \pm \frac{2x}{3} = 100$ (7)

Clearing of fractions, $3x \pm 2x = 300$ (8)

Whence, $x = \frac{300}{3 \pm 2} = 60$, or 300

Subst. in (1), $y = 100 - x = 100 - 60$, or $100 - 300$

Uniting, $y = 40$, or -200

Then, $\frac{80}{y} = \frac{80}{40} = 2$

Also, $\frac{180}{x} = \frac{180}{60} = 3$

This is one of the problems to which reference was made on p. 277 of the Algebra. The above solution

involves only pure quadratics, but by assuming a different notation, the equation after elimination becomes an affected quadratic. If only one unknown quantity be employed, or if the value of y in (1) be substituted in (4), the equation becomes $9(100 - x)^2 = 4x^2$. If this equation be expanded, instead of evolving, it reduces to $5x^2 - 1800x = -90000$, whose roots are 300 and 60, as above.

RATIO AND PROPORTION.

(ART. 329, pp. 293-295.)

3. Let	$x =$ first term.
Then,	$x : 4 :: 6 : 8$
By Theo. XI.,	$x : 4 :: 3 : 4$
By Theo. XII.,	$x : 1 :: 3 : 1$
By Theo. I.,	$x = 3$, Ans.

NOTE. We may, of course, apply Theo. I. to the first proportion, thus producing the equation $8x = 24$, which may be solved in the usual manner.

4. Given	$3 : x :: x : 1083$
By Theo. I.,	$x^2 = 3249$
Evolving,	$x = \pm 57$

5. Given	$a + x : a - x :: 11 : 7$
By Theo. VIII.,	$2a : 2x :: 18 : 4$
Dividing by 2, (Theo. XI.),	$a : x :: 9 : 2$

6. Let A and B represent the altitude and base of one triangle, and a and b the altitude and base of the other.

Then,	$B : b :: 17 : 18$
and	$A : a :: 21 : 23$
By Theo. XIII.,	$B \times A : b \times a :: 17 \times 21 : 18 \times 23$
Or,	$B \times A : b \times a :: 357 : 414$
Div. by 3, (Theo. XI.),	$B \times A : b \times a :: 119 : 138$

7. Let $x =$ no. gallons of milk.

Then $\frac{5x}{4} =$ no. gallons of mixture at first,

$\frac{5x}{4} - 3 =$ no. gallons mixture after selling,

and $\frac{5x}{4} - 3 + \frac{3}{4} = \frac{5x-9}{4} =$ no. gals. mixt. at last.

Therefore, $\frac{5x-9}{4} : \frac{5x-12}{4} :: 7 : 6$

Or (Theo. XI.), $5x-9 : 5x-12 :: 7 : 6$

• By Theo. VII., $3 : 5x-12 :: 1 : 6$

By Theo. I., $5x-12 = 18$

Transposing and uniting, $5x = 30$

Dividing by 5, $x = 6$

8. Let $x =$ A's stock,

and $y =$ B's stock.

Then, $x + 100 : y - 50 :: 4 : 3$ (1)

and $x - 50 : y + 100 :: 5 : 9$ (2)

From (1) (Theo. I.), $3x + 300 = 4y - 200$ (3)

From (2) (Theo. I.), $9x - 450 = 5y + 500$ (4)

Transp. and unit. in (3), $3x - 4y = -500$ (5)

Transp. and unit. in (4), $9x - 5y = 950$ (6)

Multiplying (5) by 3, $9x - 12y = -1500$ (7)

Subtracting (7) from (6), $7y = 2450$ (8)

Dividing by 7, $y = 350$ (9)

Substituting in (5), $3x - 1400 = -500$ (10)

Transp. and uniting, $3x = 900$ (11)

Dividing by 3, $x = 300$ (12)

9. Let x and y represent the numbers.

Then, $xy = 15$ (1)

and $x^2 + y^2 : x^2 - y^2 :: 17 : 8$ (2)

From (2), (Theo. VIII.), $2x^2 : 2y^2 :: 25 : 9$ (3)

Dividing by 2, (Theo. XI.), $x^2 : y^2 :: 25 : 9$ (4)

$$\text{Evolving (Theo. XIV.),} \quad x : y :: 5 : 3 \quad (5)$$

$$\text{Whence (Theo. I.),} \quad 3x = 5y \quad (6)$$

$$\text{Dividing by 3,} \quad x = \frac{5y}{3} \quad (7)$$

$$\text{Substituting (7) in (1),} \quad \frac{5y^2}{3} = 15 \quad (8)$$

$$\text{Whence,} \quad y^2 = 9 \quad (9)$$

$$\text{Evolving,} \quad y = \pm 3 \quad (10)$$

$$\text{Substituting in (7),} \quad x = \frac{\pm 3 \times 5}{3} = \pm 5 \quad (11)$$

$$10. \text{ Let} \quad x = \text{A's money}$$

$$y = \text{B's money,}$$

and r = ratio of the money staked to the whole amount of each.

$$\text{Then} \quad rx = \text{money staked by A,}$$

$$\text{and} \quad ry = \text{money staked by B.}$$

$$\text{Therefore,} \quad x + ry = 2(y - ry) \quad (1)$$

$$y + rx = 3(x - rx) \quad (2)$$

$$\text{and} \quad x + y = 168 \quad (3)$$

$$\text{From (1),} \quad x + 3ry = 2y \quad (4)$$

$$\text{From (2),} \quad y + 4rx = 3x \quad (5)$$

$$\text{From (4),} \quad r = \frac{2y - x}{3y} \quad (6)$$

$$\text{From (5),} \quad r = \frac{3x - y}{4x} \quad (7)$$

$$\text{Hence,} \quad \frac{3x - y}{4x} = \frac{2y - x}{3y} \quad (8)$$

$$\text{Clearing of fractions,} \quad 9xy - 3y^2 = 8xy - 4x^2 \quad (9)$$

$$\text{Or,} \quad 4x^2 + xy = 3y^2 \quad (10)$$

$$\text{Adding } 3xy, \quad 4x^2 + 4xy = 3xy + 3y^2 \quad (11)$$

$$\text{Dividing by } x + y, \quad 4x = 3y \quad (12)$$

$$\text{Or,} \quad x = \frac{3y}{4} \quad (13)$$

$$\text{Substituting in (3),} \quad \frac{3y}{4} + y = 168 \quad (14)$$

$$\text{Clearing of fractions,} \quad 3y + 4y = 672 \quad (15)$$

$$\text{Uniting terms,} \quad 7y = 672 \quad (16)$$

$$\text{Whence,} \quad y = 96 \quad (17)$$

$$\text{Substituting in (13),} \quad x = \frac{3 \times 96}{4} = 72 \quad (18)$$

$$\text{Substituting in (7),} \quad r = \frac{216 - 96}{288} = \frac{1}{2} \quad (19)$$

NOTE. If x and y be eliminated from (3), (4), and (5), we obtain the equation $12r^2 - 17r = -5$, whose roots are 1, or $\frac{1}{2}$. The former of these leads to impossible values of x and y , as it corresponds with $x + y = 0$, the factor cast out from equation (11). The factor $r - 1$, or $1 - r$, is readily discovered in the equation, and removed from it, rendering it a simple equation. Thus, after eliminating y by means of (3), (4) gives $x = \frac{112 - 168r}{1 - r}$, and (5) gives $x = \frac{42}{1 - r}$; whence, $42 = 112 - 168r$, and $r = \frac{1}{2}$.

SERIES.

ARITHMETICAL PROGRESSION.

(ART. 334, pp. 296, 297.)

1. $l = a + (n - 1)d = 5 + 19 \times 3 = 62$
2. $l = a + (n - 1)d = 4 + 29 \times 5 = 149$
3. $l = a + (n - 1)d = 10 + 4 \times (-2) = 10 - 8 = 2$
4. $l = a + (n - 1)d = -6 + 14 \times \frac{1}{2} = -6 + 7 = 1$
5. $l = a + (n - 1)d = 15 + 5 \times (-3) = 15 - 15 = 0$

(ART. 335, pp. 297, 298.)

1. $l = a + (n - 1)d = 3 + 19 \times 2 = 41$
 $S = \frac{1}{2}n(a + l) = 10 \times (3 + 41) = 10 \times 44 = 440$
2. $l = a + (n - 1)d = 7 + 5 \times (-4) = 7 - 20 = -13$
 $S = \frac{1}{2}n(a + l) = 3 \times (7 - 13) = 3 \times (-6) = -18$
3. $l = a + (n - 1)d = \frac{1}{2} + 19 \times 1 = 19\frac{1}{2}$
 $S = \frac{1}{2}n(a + l) = 10 \times (\frac{1}{2} + 19\frac{1}{2}) = 10 \times 20 = 200$

$$4. S = \frac{1}{2}n(a+l) = 10 \times (5+62) = 10 \times 67 = 670$$

$$5. l = a + (n-1)d = -3\frac{1}{2} + 20 \times \frac{1}{2} = -3\frac{1}{2} + 10 = 6\frac{1}{2}$$

$$S = \frac{1}{2}n(a+l) = \frac{1}{2} \times (-3\frac{1}{2} + 6\frac{1}{2}) = \frac{1}{2} \times (3) = \frac{3}{2} = 1\frac{1}{2}$$

(ART. 336, pp. 298-300.)

$$1. a = l - (n-1)d = 62 - 19 \times 3 = 62 - 57 = 5$$

$$2. d = \frac{l-a}{n-1} = \frac{149-4}{30-1} = \frac{145}{29} = 5$$

$$3. n = \frac{l-a}{d} + 1 = \frac{1-(-6)}{\frac{1}{2}} + 1 = 14 + 1 = 15$$

$$4. a = \frac{2S}{n} - l = \frac{2 \times 99}{9} - 19 = 22 - 19 = 3$$

$$5. d = \frac{l-a}{n-1} = \frac{2-10}{5-1} = \frac{-8}{4} = -2$$

(ART. 337, pp. 300, 301.)

$$1. \frac{a+l}{2} = \frac{6+20}{2} = \frac{26}{2} = 13$$

$$2. d = \frac{l-a}{n-1} = \frac{14-5}{4-1} = \frac{9}{3} = 3$$

$$a+d = 5+3 = 8; 8+3 = 11.$$

$$3. \frac{a+l}{2} = \frac{\frac{1}{2} + \frac{1}{2}}{2} = \frac{\frac{1+1}{2}}{2} = \frac{1}{2}$$

$$4. d = \frac{l-a}{n-1} = \frac{3-1}{5-1} = \frac{2}{4} = \frac{1}{2}$$

$$a+d = 1 + \frac{1}{2} = 1\frac{1}{2}; 1\frac{1}{2} + \frac{1}{2} = 2; 2 + \frac{1}{2} = 2\frac{1}{2}$$

$$5. \frac{\frac{a}{2} + \frac{b}{2}}{2} = \frac{a}{4} + \frac{b}{4} = \frac{1}{4}(a+b)$$

$$6. d = \frac{l-a}{n-1} = \frac{\frac{11}{2} - \frac{1}{2}}{4-1} = \frac{\frac{10}{2}}{3} = \frac{5}{3}$$

$$a+d = \frac{1}{3} + \frac{5}{3} = \frac{6}{3} = 2; \frac{5}{3} + \frac{1}{3} = \frac{6}{3} = 2$$

$$7. \quad d = \frac{l-a}{n-1} = \frac{y-x}{4-1} = \frac{y-x}{3}$$

$$a+d = x + \frac{y-x}{3} = \frac{2x+y}{3}; \quad \frac{2x+y}{3} + \frac{y-x}{3} = \frac{x+2y}{3}$$

NOTE. It will be seen that the common difference is found by dividing the difference between the two given terms by the required number of means to be inserted, plus 1; that is, $d = \frac{l-a}{m+1}$, in which m represents the number of means.

PROBLEMS.

(ART. 338, pp. 302, 303.)

$$1. \quad S = \frac{1}{2}n(a+l) = 6 \times (1+12) = 6 \times 13 = 78$$

$$2. \quad l = a + (n-1)d = \frac{1}{3} + 14 \times \frac{1}{3} = \frac{1}{3} + \frac{14}{3} = \frac{15}{3} = 5$$

$$3. \quad l = a + (n-1)d = 15 + 19 \times (-4) = 15 - 76 = -61$$

$$S = \frac{1}{2}n(a+l) = 10 \times (15-61) = 10 \times (-46) = -460$$

$$4. \quad l = a + (n-1)d = 2 + 24 \times 3 = 2 + 72 = 74$$

$$S = \frac{1}{2}n(a+l) = \frac{25}{2} \times (2+74) = \frac{25}{2} \times 76 = 950$$

$$5. \quad d = \frac{l-a}{n-1} = \frac{-\frac{1}{2}-\frac{1}{6}}{7-1} = -\frac{1}{6}$$

$$a+d = \frac{1}{2}-\frac{1}{6} = \frac{2}{6} = \frac{1}{3}; \quad \frac{2}{6}-\frac{1}{6} = \frac{1}{6}; \quad \frac{1}{6}-\frac{1}{6} = 0; \quad 0-\frac{1}{6} = -\frac{1}{6};$$

$$-\frac{1}{6}-\frac{1}{6} = -\frac{2}{6} = -\frac{1}{3}.$$

6. The values of n from the two fundamental formulas are $n = \frac{l-a}{d} + 1 = \frac{2S}{l+a}$, whence $d = \frac{l^2-a^2}{2S-l-a}$

$$= \frac{2500-1}{408-50-1} = \frac{2499}{357} = 7.$$

$$\text{Or, } n = \frac{2S}{l+a} = \frac{2 \times 204}{50+1} = \frac{408}{51} = 8$$

$$d = \frac{l-a}{n-1} = \frac{50-1}{8-1} = \frac{49}{7} = 7$$

$$7. \quad \text{Art. 336, (8), } n = \frac{2S}{l+a} = \frac{2 \times 25}{7+3} = \frac{50}{10} = 5$$

8. The common difference is the interest of \$20 for one year, at 4 per cent, or $20 \times \frac{4}{100} = \frac{4}{5}$.

$$l = a + (n - 1)d = 20 + 19 \times \frac{4}{5} = 20 + 15\frac{2}{5} = 35\frac{2}{5}$$

$$S = \frac{1}{2}n(a + l) = 10 \times (20 + 35\frac{2}{5}) = 10 \times 55\frac{2}{5} = 552$$

NOTE. The amount of the last year's savings is simply \$20; but, as it is smaller than the amount of the first year's savings, which has 19 years' interest added to it, we use the smallest as a , or the first term, and the largest as l , or the last term.

9. Art. 336 (4), $a = \frac{2S}{n} - l = \frac{2 \times 140}{8} - 7 = 35 - 7 = 28$

Art. 336 (5), $d = \frac{l - a}{n - 1} = \frac{7 - 28}{8 - 1} = \frac{-21}{7} = -3$

$a + d = 28 - 3 = 25$; $25 - 3 = 22$, etc.

10. $l = a + (n - 1)d = 2000 + (x - 1)\frac{2000}{10} = 2000\left(1 + \frac{x - 1}{10}\right)$

$$S = \frac{1}{2}n(a + l) = \frac{x}{2} \left[2000 + 2000\left(1 + \frac{x - 1}{10}\right) \right] = \frac{2000x}{2} \left(2 + \frac{x - 1}{10} \right)$$

$$= 2000x + 100x(x - 1) = 100x^2 + 1900x$$

From the conditions, $\frac{100x^2 + 1900x}{10} = 3300$

Or, $10x^2 + 190x = 3300$

Dividing by 10, $x^2 + 19x = 330$

Completing square, $x^2 + 19x + \left(\frac{19}{2}\right)^2 = \frac{1681}{4}$

Evolving, $x + \frac{19}{2} = \pm \frac{41}{2}$

Whence, $x = 11$, or -30

11. Let x = the number of seconds.

Then $8x$ = distance run by the hare.

$l = a + (n - 1)d = 8 + (x - 1) \times \frac{1}{2} = \frac{x + 15}{2}$ = distance run by the greyhound during the last second.

$S = \frac{n}{2}(a + l) = \frac{x}{2} \left(8 + \frac{x + 15}{2} \right) = \frac{x^2 + 31x}{4}$ = whole distance run by the greyhound.

From the conditions, $\frac{x^2 + 31x}{4} = 8x + 60$

Clearing of fractions, $x^2 + 31x = 32x + 240$

Whence,

$$x^2 - x = 240$$

Completing square,

$$x^2 - x + \frac{1}{4} = 240\frac{1}{4} = 240\frac{1}{4}$$

Evolving,

$$x - \frac{1}{2} = \pm \sqrt{240\frac{1}{4}}$$

Whence,

$$x = 16, \text{ or } -15$$

The negative root of the equation, 15, is the answer to a problem whose conditions are the reverse of those given; that is, both must run in an opposite direction, the hare must overtake the greyhound, and the distances run by the greyhound in successive seconds must be a continuation of the given series in the other direction, $7\frac{1}{2}$ feet the first second, 7 feet the second, $6\frac{1}{2}$ the third, etc.

12. Let

$x =$ first term,

and

$y =$ common difference.

Then $x, x + y, x + 2y, x + 3y$, etc., is the series.

Therefore,

$$2x + y = 2\frac{1}{2} \quad (1)$$

and

$$x + 3y = 2\frac{1}{2} \quad (2)$$

Multiplying (2) by 2,

$$2x + 6y = 5 \quad (3)$$

Subtracting (1) from (3),

$$5y = \frac{1}{2} \quad (4)$$

Dividing (4) by 5,

$$y = \frac{1}{10} \quad (5)$$

Substituting in (2),

$$x + \frac{3}{10} = \frac{5}{10}$$

Whence,

$$x = 1$$

Substituting the value of x and y in the series $x, x + y, x + 2y, x + 3y$, we obtain 1, $1\frac{1}{10}$, 2, $2\frac{1}{10}$.

NOTE. It is sometimes more convenient to use $x - y, x$, and $x + y$ for three terms of an arithmetical series, and $x - 3y, x - y, x + y$, and $x + 3y$ for four terms.

13. In this problem, $a = 16.1$, $d = 32.2$, and $n = 20$; hence the distance through which a body would fall in the twentieth second $= l = a + (n - 1)d = 16.1 + 19 \times 32.2 = 16.1 + 611.8 = 627.9$. The distance through which it would fall in 20 seconds $= S = \frac{1}{2}n(a + l) = 10 \times (16.1 + 627.9) = 10 \times 644 = 6440$. But $6440 \text{ ft.} = \frac{6440}{5280} \text{ miles} = 1.22 \text{ m.}$, nearly.

GEOMETRICAL PROGRESSION.

(ART. 343, p. 305.)

1. $l = ar^{n-1} = 5 \times 4^7 = 5 \times 16384 = 81920.$
2. $l = ar^{n-1} = 28672 \times (\frac{1}{4})^6 = \frac{28672}{4096} = 7.$
3. $l = ar^{n-1} = 5 \times 4^8 = 5 \times 65536 = 327680.$
4. $l = ar^{n-1} = 100 \times (\frac{2}{3})^5 = 100 \times \frac{32}{243} = \frac{3200}{243} = 13 \frac{41}{243}.$
5. $l = ar^{n-1} = 3 \times 2^8 = 3 \times 256 = 768.$

(ART. 344, pp. 306, 307.)

1. $S = \frac{rl - a}{r - 1} = \frac{2 \times 1024 - 1}{2 - 1} = 2048 - 1 = 2047.$
2. $S = \frac{a(r^n - 1)}{r - 1} = \frac{6(4^8 - 1)}{4 - 1} = \frac{6 \times 65535}{3} = 131070.$
3. $S = \frac{rl - a}{r - 1} = \frac{\frac{1}{3} \times \frac{1}{162} - \frac{1}{2}}{\frac{1}{3} - 1} = \frac{-\frac{113}{162}}{-\frac{2}{3}} = 1\frac{113}{54}.$
4. $S = \frac{ar^n - a}{r - 1} = \frac{1 \times 3^{12} - 1}{3 - 1} = \frac{531441 - 1}{3 - 1} = \frac{531440}{2} = 265720.$
5. $S = \frac{a(r^n - 1)}{r - 1} = \frac{4[(\frac{1}{2})^{16} - 1]}{\frac{1}{2} - 1} = \frac{4(1 - \frac{1}{2^{16}})}{\frac{1}{2}} = 8(1 - \frac{1}{2^{16}})$
 $= 8 - \frac{8}{2^{16}} = 8 - \frac{2^3}{2^{16}} = 8 - \frac{1}{2^{13}} = 8 - \frac{1}{8192}.$

(ART. 345, pp. 307, 308.)

1. $S = \frac{a}{1 - r} = \frac{4}{1 - \frac{1}{2}} = \frac{4}{\frac{1}{2}} = 8.$
2. $S = \frac{a}{1 - r} = \frac{\frac{7}{8}}{1 - \frac{3}{8}} = \frac{\frac{7}{8}}{\frac{5}{8}} = \frac{7}{5}.$
3. $S = \frac{a}{1 - r} = \frac{\frac{79}{100}}{1 - \frac{1}{100}} = \frac{\frac{79}{100}}{\frac{99}{100}} = \frac{79}{99}.$

$$4. S = \frac{a}{1-r} = \frac{1}{1-\frac{1}{10}} = \frac{1}{\frac{9}{10}} = \frac{10}{9} = 1\frac{1}{9}.$$

$$5. S = \frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} \times \frac{x}{x} = \frac{x}{x-1}.$$

$$6. S = \frac{a}{1-r} = \frac{1}{1+\frac{1}{2}} = \frac{1}{\frac{3}{2}} = \frac{2}{3}.$$

(ART. 346, pp. 308, 309.)

$$1. a = \frac{l}{r^{n-1}} = \frac{405}{3^4} = \frac{405}{81} = 5.$$

$$2. r = \left(\frac{l}{a}\right)^{\frac{1}{n-1}} = \left(\frac{768}{3}\right)^{\frac{1}{3}} = (256)^{\frac{1}{3}} = 2.$$

$$3. r = \frac{S-a}{S-l} = \frac{9555-7168}{9555-7} = \frac{2387}{9548} = \frac{1}{4}.$$

$$4. a = rl - (r-1)S = 2 \times 3072 - (2-1) \times 6141 \\ = 6144 - 6141 = 3.$$

5. Formula (4), by a simple transposition of terms and change of signs, becomes, after dividing by r ,

$$l = \frac{a + (r-1)S}{r} = \frac{2 + (3-1) \times 6560}{3} = \frac{2 + 13120}{3} = \frac{13122}{3} \\ = 4374.$$

NOTE. Formula (4) is simply (2) cleared of fractions, etc.; hence the solution of this example is based entirely upon the second fundamental formula.

(ART. 347, pp. 309-311.)

$$1. \sqrt{\frac{1}{4} \times \frac{3}{4}} = \sqrt{\frac{1}{4}} = \frac{1}{2}.$$

$$2. r = \left(\frac{l}{a}\right)^{\frac{1}{m+1}} = \left(\frac{320}{5}\right)^{\frac{1}{3}} = (64)^{\frac{1}{3}} = 4; \\ ar = 5 \times 4 = 20; ar^2 = 20 \times 4 = 80.$$

$$3. \sqrt{30 \times 7\frac{1}{2}} = \sqrt{225} = 15.$$

$$4. \quad r = \left(\frac{l}{a}\right)^{\frac{1}{m+1}} = \left(\frac{486}{6}\right)^{\frac{1}{4}} = (81)^{\frac{1}{4}} = 3;$$

$$ar = 6 \times 3 = 18; ar^2 = 18 \times 3 = 54; ar^3 = 54 \times 3 = 162.$$

$$5. \quad \text{Arith. mean} = \frac{1 + \frac{1}{9}}{2} = \frac{\frac{10}{9}}{2} = \frac{5}{9}.$$

$$\text{Geom. mean} = \sqrt{1 \times \frac{1}{9}} = \sqrt{\frac{1}{9}} = \frac{1}{3}.$$

$$\frac{5}{9} - \frac{1}{3} = \frac{5}{9} - \frac{3}{9} = \frac{2}{9}$$

NOTE. If it is preferred, the formula $r = \left(\frac{l}{a}\right)^{\frac{1}{n-1}}$ may be used in place of $r = \left(\frac{l}{a}\right)^{\frac{1}{m+1}}$, or the first part of the rule, remembering that n , the number of terms, is always 2 greater than m , the number of means.

PROBLEMS.

(ART. 348, pp. 311, 312.)

$$1. \quad a = \frac{1}{3}, ar = 1; r = \frac{ar}{a} = \frac{1}{\frac{1}{3}} = 3;$$

$$ar^2 = 1 \times 3 = 3; ar^3 = 3 \times 3 = 9.$$

$$2. \quad \frac{ar^4}{ar^2} = r^2 = \frac{300}{75} = 4; r = \sqrt{4} = 2; 300 \times 2 = 600.$$

$$3. \quad l = ar^{n-1} = 1 \times 2^9 = 512.$$

$$4. \quad \text{Formula (4), Art. 344, } S = \frac{a(r^n - 1)}{r - 1}, \text{ after clearing the equation of fractions and dividing by } r^n - 1, \text{ becomes } a = \frac{(r-1)S}{r^n - 1} = \frac{(\frac{3}{2} - 1) \times 102.95}{(\frac{3}{2})^7 - 1} = \frac{\frac{1}{2} \times 102.95}{\frac{2187}{128} - 1} = \frac{51.475}{\frac{2059}{128}} = .025 \times 128 = 3.20.$$

$$5. \quad \text{Formula (5), Art. 346, } r = \left(\frac{l}{a}\right)^{\frac{1}{n-1}} = \left(\frac{2048}{1}\right)^{\frac{1}{11}} = (2048)^{\frac{1}{11}} = 2.$$

$$6. \quad \text{Formula, Art. 345, } S = \frac{a}{1-r} = \frac{20}{1-\frac{1}{2}} = \frac{20}{\frac{1}{2}} = 400.$$

7. Let $x =$ a side of the square,
and $x^2 =$ its area.

Then, $x^2 = 18 \times 288$

Evolving, $x = \sqrt{18 \times 288} = \sqrt{5184} = 72$

The side of the square is then a geometrical mean between the sides of the rectangle. This mean is obtained as in Art. 347.

8. $r^2 = \frac{ar^3}{ar} = \frac{54}{2} = 9$, and $r = \sqrt{9} = 3$;

$a = \frac{ar}{r} = \frac{6}{3} = 2$.

9. By Art. 347, $r = \left(\frac{l}{a}\right)^{\frac{1}{n-1}} = \left(\frac{128}{1}\right)^{\frac{1}{7-1}} = (128)^{\frac{1}{6}} = 2$;

$a = 1$; $ar = 1 \times 2 = 2$; $ar^2 = 2 \times 2 = 4$;
 $ar^3 = 4 \times 2 = 8$; $ar^4 = 8 \times 2 = 16$; $ar^5 = 16 \times 2 = 32$;
 $ar^6 = 32 \times 2 = 64$; $ar^7 = 64 \times 2 = 128$.

10. $a = \frac{x}{y}$, $ar = x$, and $ar^2 = xy$;

$r = \frac{ar^2}{ar} = \frac{xy}{x} = y$; or, $r = \frac{ar}{a} = x \div \frac{x}{y} = y$.

11. Let $x =$ the first term,
and $y =$ the ratio.

Then x , xy , and xy^2 represent the three shares.

Hence, $x + xy + xy^2 = 210$ (1)

and $x - xy^2 = 90$ (2)

From (1), $x = \frac{210}{1 + y + y^2}$ (3)

From (2), $x = \frac{90}{1 - y^2}$ (4)

Hence, $\frac{90}{1 - y^2} = \frac{210}{1 + y + y^2}$ (5)

Dividing by 30, $\frac{3}{1 - y^2} = \frac{7}{1 + y + y^2}$ (6)

Clearing of fractions, $3 + 3y + 3y^2 = 7 - 7y^2$ (7)

Transp. and uniting, $10y^2 + 3y = 4$ (8)

Dividing by 10, $y^2 + \frac{3}{10}y = \frac{4}{10}$ (9)

Completing square, $y^2 + \frac{3}{10}y + \frac{9}{400} = \frac{169}{400}$ (10)

Evolving, $y + \frac{3}{20} = \pm \frac{13}{20}$ (11)

Whence, $y = \frac{1}{2}$, or $-\frac{1}{2}$

Substituting in (4), $x = \frac{90}{1 - \frac{1}{2}} = \frac{90}{\frac{1}{2}} = 180$

Hence, $xy = 120 \times \frac{1}{2} = 60$

and $xy^2 = 120 \times \frac{1}{4} = 30$

12. Let x, xy, xy^2, xy^3 , etc., represent the series of terms,

Then, $x + xy = 1\frac{1}{2}$ (1)

and $xy^2 + xy^3 = 12$ (2)

Dividing (2) by (1), $y^2 = 9$ (3)

Evolving, $y = \pm 3$ (4)

Substituting in (1), $x + 3x = \frac{3}{2}$ (5)

Whence, $4x = \frac{3}{2}$ (6)

Or, $x = \frac{3}{8}$ (7)

Therefore, $xy = \frac{3}{8} \times 3 = \frac{9}{8}$

$xy^2 = \frac{3}{8} \times 9 = \frac{27}{8}$

and $xy^3 = \frac{3}{8} \times 27 = \frac{81}{8}$

If $y = -3$, then $x = -\frac{3}{8}$, and the series $-\frac{3}{8}, 2, -6, 18$ etc., will satisfy the conditions of the problem.

13. Let x, xy , and xy^2 , represent the three terms.

Then, $x + xy + xy^2 = 35$ (1)

and $xy : xy^2 - x :: 2 : 3$ (2)

From (2), $3xy = 2xy^2 - 2x$ (3)

Dividing by x , $3y = 2y^2 - 2$ (4)

Or, $2y^2 - 3y = 2$ (5)

Multiplying by 8, $16y^2 - 24y = 16$ (6)

Completing square, $16y^2 - 24y + 9 = 25$ (7)

Evolving, $4y - 3 = \pm 5$ (8)

Whence, $4y = 8$, or -2

Dividing by 4, $y = 2$, or $-\frac{1}{2}$

Substituting in (1), $x + 2x + 4x = 35$

Uniting terms, $7x = 35$

Dividing by 7,

$$x = 5$$

Therefore,

$$xy = 5 \times 2 = 10$$

and

$$xy^2 = 5 \times 4 = 20$$

If $-\frac{1}{2}$, the second value of y , be employed, we obtain the equation $x - \frac{x}{2} + \frac{x}{4} = 35$, from which we derive $x = 46\frac{2}{3}$.

Hence the series $46\frac{2}{3}$, $-23\frac{1}{3}$, and $11\frac{2}{3}$ will also satisfy the conditions of the problem, algebraically.

14. Let x , xy , xy^2 , and xy^3 represent the numbers.

$$\text{Then,} \quad xy^3 - xy = 24 \quad (1)$$

$$\text{and} \quad x + xy^3 : xy + xy^2 :: 7 : 3 \quad (2)$$

$$\text{From (2),} \quad 3(x + xy^3) = 7(xy + xy^2) \quad (3)$$

$$\text{Dividing by } x + xy, \quad 3(1 - y + y^2) = 7y \quad (4)$$

$$\text{Whence,} \quad 3y^2 - 10y = -3 \quad (5)$$

$$\text{Multiplying by 3,} \quad 9y^2 - 30y = -9 \quad (6)$$

$$\text{Completing square,} \quad 9y^2 - 30y + 25 = 16 \quad (7)$$

$$\text{Evolving,} \quad 3y - 5 = \pm 4 \quad (8)$$

$$\text{Whence,} \quad 3y = 9, \text{ or } 1$$

$$\text{Dividing by 3,} \quad y = 3, \text{ or } \frac{1}{3}$$

$$\text{From (1),} \quad x = \frac{24}{y^3 - y}$$

$$\text{Substituting values of } y, \quad x = \frac{24}{27 - 3}, \text{ or } \frac{24}{\frac{1}{27} - \frac{1}{3}}$$

$$\text{Reducing,} \quad x = 1, \text{ or } -81$$

$$xy = 1 \times 3, \text{ or } -81 \times \frac{1}{3} = 3, \text{ or } -27$$

$$xy^2 = 1 \times 9, \text{ or } -81 \times \frac{1}{9} = 9, \text{ or } -9$$

$$xy^3 = 1 \times 27, \text{ or } -81 \times \frac{1}{27} = 27, \text{ or } -3$$

It will be seen that the series -81 , -27 , -9 , and -3 will answer the conditions of the problem, algebraically, as well as 1 , 3 , 9 , 27 .

NOTE. Instead of x , xy , xy^2 , &c., it is sometimes convenient to use x^2 , xy , y^2 ; or x , $\sqrt{xy}$, y ; or $\frac{x}{y}$, x , xy , when there are three unknown quantities in geometrical progression; or $\frac{x^2}{y}$, x , y , $\frac{y^2}{x}$, when there are four.

MISCELLANEOUS EXAMPLES.

(Pages 313-324.)

$$1. \quad ax \times 2by \times z = 2abxyz.$$

$$2. \quad 2a^2 - b^2 + x^2 = 2 \times 16 - 36 + 4 = 36 - 36 = 0.$$

$$3. \quad \frac{2x^2 - ax}{a^2 - x^2} = \frac{2 \times \frac{1}{4} - 2 \times (-\frac{1}{2})}{8 - (-\frac{1}{8})} = \frac{\frac{1}{2} + \frac{2}{2}}{8 + \frac{1}{8}} = \frac{\frac{3}{2}}{\frac{65}{8}} = \frac{12}{65}.$$

$$4. \quad \begin{array}{r} a^2 + ab + b^2 \\ a^2 - ab + b^2 \\ \hline a^4 + a^3b + a^2b^2 \\ - a^3b - a^2b^2 - ab^3 \\ \hline a^2b^2 + ab^3 + b^4 \\ \hline a^4 + a^2b^2 + b^4 \end{array}$$

$$5. \quad 48a^2b^3x^3 = 2^4 \times 3a^2b^3x^3 = 2 \times 2 \times 2 \times 2 \times 3a^2b^3x^3.$$

$$6. \quad 16a^2b^2 - 9x^2 = (4ab + 3x)(4ab - 3x) \text{ by Art. 91.}$$

$$7. \quad 1 - x + \frac{x^2}{1+x} = \frac{(1-x)(1+x)}{1+x} + \frac{x^2}{1+x} = \frac{1-x^2+x^2}{1+x} = \frac{1}{1+x}.$$

$$8. \quad \frac{5a+3x}{3} - (a-x) = \frac{5a+3x-3a+3x}{3} = \frac{2a+6x}{3}.$$

$$9. \quad (x^2 + 2x - 3) \div (x - 1) = x + 3.$$

$$10. \quad \frac{a+1}{a-1} \times (a-1) = a+1, \text{ Ans.}$$

$$11. \quad x - \frac{3x}{1+x} = \frac{x+x^2-3x}{1+x} = \frac{x^2-2x}{1+x};$$

$$\frac{x^2-2x}{1+x} \div \frac{-2x}{1+x} = \frac{x^2-2x}{1+x} \times \frac{1+x}{-2x} = \frac{x^2-2x}{-2x} = \frac{x-2}{-2} = \frac{2-x}{2}.$$

$$12. \quad 16x^3y^4 = 2 \times 2 \times 2 \times 2xyyy$$

$$28axy = 2 \times 2 \times 7axy$$

$$4xy = 2 \times 2xy$$

13. Let $x =$ number of apple-trees,
and $3x =$ number of pear-trees.

$$\text{Then, } 4(x-4) = 3x - 4$$

$$\text{Or, } 4x - 16 = 3x - 4$$

$$\text{Whence, } x = 12$$

$$\text{Then, } 3x = 36$$

14. Let $x =$ each son's share,
 $y =$ each daughter's share,
and $x + y =$ the widow's share.

$$\text{Then, } x + y + 2x + 3y = 1700 \quad (1)$$

$$\text{and } 3y = 2x \quad (2)$$

$$\text{From (2), } y = \frac{2x}{3} \quad (3)$$

$$\text{Substituting (3) in (1), } x + \frac{2x}{3} + 2x + 2x = 1700 \quad (4)$$

$$\text{Or, } \frac{17x}{3} = 1700 \quad (5)$$

$$\text{Dividing by } \frac{17}{3}, \quad x = 300 \quad (6)$$

$$\text{Substituting in (3), } y = \frac{2 \times 300}{3} = 200 \quad (7)$$

$$\text{Adding (6) and (7), } x + y = 500 \quad (8)$$

15. Let $x =$ no. gallons per stroke, of the second,
 $\frac{2}{3} =$ no. strokes of the first, per minute,
and $\frac{2}{3} =$ no. strokes of the second, per minute.

$$\text{Then, } x \times \frac{2}{3} = 2 \times \frac{2}{3}$$

$$\text{Or, } \frac{2x}{3} = 2$$

$$\text{Clearing of fractions, } 2x = 6$$

$$\text{Whence, } x = 3$$

16. Let $x =$ the distance, in miles.
Then $\frac{x}{192000} =$ time for light to travel x miles,
and $\frac{x}{\frac{1090}{5280}} = \frac{528x}{109} =$ time for sound to travel x miles.

From the conditions, $\frac{528x}{109} - \frac{x}{192000} = 10\frac{1}{2}$

Clearing of fractions, $101376000x - 109x = 219744000$

Uniting terms, $101375891x = 219744000$

Dividing by 101375891, $x = 2\frac{1}{8}$, nearly.

NOTE. As light travels with such great velocity, it is customary to consider its motion instantaneous, for terrestrial distances. If, then, we neglect the time required for light to pass over x miles, we obtain

$$\frac{528x}{109} = 10\frac{1}{2}, \text{ or } 528x = 1144.5,$$

whence $x = \frac{1144.5}{528} = 2\frac{177}{1056} = 2\frac{59}{352} = 2\frac{1}{8}$, nearly.

If we had used a for 192000, b for $\frac{1090}{5280}$, and c for $10\frac{1}{2}$, we should have obtained $x = \frac{abc}{a-b}$. As b is so small in comparison with a , the error would be very small if we regard a as cancelling $a-b$, thus reducing the expression to $x = bc$.

17. Let $x =$ distance traveled.

Then $8 - x =$ distance from the town,

$\frac{x}{4} =$ time of the first, in hours,

and $\frac{x}{4\frac{1}{2}} = \frac{2x}{9} =$ time of the second, in hours.

Also $10 \text{ minutes} = \frac{1}{6} \text{ hour}$

From the conditions, $\frac{x}{4} - \frac{2x}{9} = \frac{1}{6}$

Clearing of fractions, $9x - 8x = 6$

Or, $x = 6$

Whence, $8 - x = 8 - 6 = 2$

18. Let $x =$ the distance.

Then $\frac{x}{10} =$ time on the coach,

and $\frac{x}{4} =$ time in returning.

Therefore, $\frac{x}{4} + \frac{x}{10} = 1\frac{1}{2}$

Clearing of fractions, $5x + 2x = 30$
 Or, $7x = 30$
 Dividing by 7, $x = 4\frac{2}{7}$, Ans.

19. Let $x =$ the price per dozen, in pence,
 and $y =$ the number of coins of each kind.
 Then $2y =$ number of dozens of oranges,
 $2xy =$ the price of the whole,
 $12y =$ the value of the shillings, in pence,
 and $\frac{y}{2} =$ the value of the halfpence, in pence.

Therefore, $2xy = 12y + \frac{y}{2}$
 Dividing by y , $2x = 12 + \frac{1}{2}$
 Dividing by 2, $x = 2\frac{1}{4} + \frac{1}{4} = 2\frac{1}{2} = 6\frac{1}{2}$

20. As the clock *gains*, it is evident the true time is *before* mid-day.

Let $x =$ the true time before Wednesday noon,
 and $a = 1440$ minutes $= 24$ hours.

Then $a - x =$ the true time after Tuesday noon.

As the clock gains 1 minute in 24 hours, or a minutes, it indicates $a + 1$ minutes to every a minutes of true time. At the required time, the clock indicates a minutes.

Hence, $a : a - x :: a + 1 : a$.

Mult. extremes and means, $a^2 = (a - x)(a + 1)$

Or, $a^2 = a^2 + a - x(a + 1)$

Whence, $(a + 1)x = a$

Dividing by $a + 1$, $x = \frac{a}{a + 1} = \frac{1440}{1441}$

NOTE. The above proportion expressed in numbers would be

cl. time	tr. time	cl. time	tr. time
1440	: 1440 - x	::	1441 : 1440

21. Let $x =$ the portion at 5 per cent,
 $a = 100000 =$ the whole principal,
 and $b = 4290 =$ the whole income.
 Then $a - x =$ the portion at 4 per cent.

Therefore, $\frac{5x}{100} + \frac{4(a-x)}{100} = b$

Clearing of fractions, $5x + 4a - 4x = 100b$

Whence, $x = 100b - 4a = 429000 - 400000$

Or, $x = 29000$

22. Let $x =$ the number bought.

Then $\frac{180}{x} =$ the cost of each,

$\frac{180}{x-2} =$ the sum received for each,

and $\frac{180}{x} \times \frac{4}{3} = \frac{240}{x} =$ per centage gained on each.

Therefore, $\frac{180}{x-2} = \frac{180}{x} + \frac{180}{x} \times \frac{240}{100x}$

Dividing by 180, $\frac{1}{x-2} = \frac{1}{x} + \frac{12}{5x^2}$

Clearing of fractions, $5x^2 = 5x^2 - 10x + 12x - 24$

Whence, $2x = 24$

Dividing by 2, $x = 12$

23. Let $x =$ sum left for each.

Then, $4x = 5(x - 1000)$

Or, $4x = 5x - 5000$

Whence, $x = 5000$, Ans.

24. Let $x =$ no. ounces of gold,
and $4160 - x =$ no. ounces of silver.

Then $\frac{x}{19\frac{1}{4}} = \frac{4x}{77} =$ weight of water displaced by gold,
and

$\frac{4160 - x}{10\frac{1}{2}} = \frac{8320 - 2x}{21} =$ weight of water displaced by silver.

Therefore, $\frac{4x}{77} + \frac{8320 - 2x}{21} = 250$

Clear. of fract., $12x + 91520 - 22x = 57750$

Whence, $10x = 33770$

Dividing by 10,

$$x = 3377$$

Then,

$$4160 - x = 4160 - 3377 = 783$$

25. Let

 $x =$ daily wages,

and

 $y =$ cost of board, per day.

Then,

$$12x - 17y = 35 \quad (1)$$

and

$$16x - 23y = 33 \quad (2)$$

Multiplying (1) by 4,

$$48x - 68y = 140 \quad (3)$$

Multiplying (2) by 3,

$$48x - 69y = 99 \quad (4)$$

Subtracting (4) from (3),

$$y = 41 \quad (5)$$

Substituting in (1),

$$12x - 697 = 35 \quad (6)$$

Whence,

$$12x = 732 \quad (7)$$

Dividing by 12,

$$x = 61 \quad (8)$$

It will be seen that the daily wages were 20 cents and board; but 41 cents are to be deducted for every day he was idle.

26. Let

 $x =$ the number.

Then,

$$\frac{2x}{3} + \frac{x}{4} - 2 = \frac{11x}{12} + 3$$

Clearing of fractions,

$$8x + 3x - 24 = 11x + 36$$

Whence,

$$0x = 60$$

Dividing by 0,

$$x = \frac{60}{0} = \infty$$

$$27. (a^{\frac{3}{2}} - b^{\frac{5}{2}})(a^{\frac{3}{2}} + b^{\frac{5}{2}}) = a^3 - b^5. \text{ Ans. } a^{\frac{3}{2}} + b^{\frac{5}{2}}.$$

$$28. \frac{y - x + \frac{a}{2}}{7\frac{3}{4}} \times \frac{4}{4} = \frac{4y - 4x + 2a}{31}.$$

$$29. \quad 4(1 - x^2) = 2^3(1 + x)(1 - x)$$

$$8(1 - x) = 2^3(1 - x)$$

$$4(1 + x^2) = 2^2(1 + x^2)$$

$$2^3(1 + x)(1 - x)(1 + x^2)$$

$$= 2^3(1 - x^2)(1 + x^2) = 8(1 - x^4)$$

30. The repeating decimal .999, etc. forms an infinite

descending series, in which the first term is $\frac{9}{10}$ and the ratio $\frac{1}{10}$. Hence

$$S = \frac{a}{1-r} = \frac{\frac{9}{10}}{1-\frac{1}{10}} = \frac{\frac{9}{10}}{\frac{9}{10}} = 1$$

31. Let x = time required by B,
and y = time required by C.

Then $\frac{1}{20}$, $\frac{1}{x}$, and $\frac{1}{y}$ are the parts each would do in 1 day.

From the conditions,
$$\frac{1}{x} + \frac{1}{y} = \frac{1}{12} \quad (1)$$

Also,
$$\frac{6}{20} + \frac{6}{x} + \frac{6}{y} + \frac{3}{y} = 1 \quad (2)$$

From (2),
$$\frac{6}{x} + \frac{9}{y} = \frac{14}{20} \quad (3)$$

Or,
$$\frac{2}{x} + \frac{3}{y} = \frac{7}{30} \quad (4)$$

Multiplying (1) by 2,
$$\frac{2}{x} + \frac{2}{y} = \frac{1}{6} \quad (5)$$

Subtracting (5) from (4),
$$\frac{1}{y} = \frac{1}{15} \quad (6)$$

Whence,
$$y = 15 \quad (7)$$

Substituting in (1),
$$\frac{1}{x} + \frac{1}{15} = \frac{1}{12} \quad (8)$$

Or,
$$\frac{1}{x} = \frac{1}{12} - \frac{1}{15} = \frac{1}{60} \quad (9)$$

Whence,
$$x = 60 \quad (10)$$

NOTE. Negative exponents may, of course, be used in place of the fractional forms given above.

32. $10000 = 100 \times 100 = 4 \times 25 \times 4 \times 25.$

33. $(-2a^{\frac{2}{3}})^7 = -2^7 a^{\frac{14}{3}} = -128 a^{\frac{2}{3}}.$

34. Let x = the length of the pile.

Then,
$$\frac{x}{5} + \frac{3x}{7} + 13 = x$$

Clearing of fractions, $7x + 15x + 455 = 35x$

Whence,

$$13x = 455$$

Dividing by 13,

$$x = 35, \text{ Ans.}$$

35. Let $x = A$'s age at his death.

Then the gift of the year of A 's death must have been $4x$, that of the year previous $4x - 4$, and so on, the sums forming an arithmetical progression, of which the number of terms is 8, the common difference 4, and the last term $4x$.

$$a = l - (n - 1)d = 4x - 7 \times 4 = 4x - 28$$

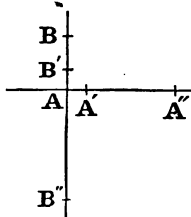
$$S = \frac{1}{2}n(a + l) = 4 \times (4x - 28 + 4x) = 32x - 112$$

$$\text{Therefore, } 32x - 112 = 1904$$

$$\text{Or, } 32x = 2016$$

$$\text{Whence, } x = 63$$

The year of his death must have been 7 years later than 1855, or 1862, and as his age was then 63 years, he must have been born in $1862 - 63 = 1799$.



36. Let the points A and B of the accompanying figure represent the positions of A and B at the first time mentioned; A' and B' their positions 2 minutes afterwards; and A'' and B'' their positions in 8 minutes more.

Let

$x = A$'s speed per minute,

and

$y = B$'s speed per minute.

Then

$2x =$ the distance AA' ,

$2y =$ the distance BB' ,

$10x =$ the distance AA'' ,

$10y =$ the distance BB'' .

and

Also,

$540 =$ the distance AB .

From the conditions of the problem, $AA' = AB' = AB$
— BB' , and $AA'' = AB'' = BB'' - AB$.

Hence,

$$2x = 540 - 2y \quad (1)$$

and

$$10x = 10y - 540 \quad (2)$$

From (1), $x + y = 270$ (3)

From (2), $x - y = -54$ (4)

Adding (3) and (4), $2x = 216$ (5)

Whence, $x = 108$ (6)

Subtracting (4) from (3), $2y = 324$ (7)

Whence, $y = 162$ (8)

37. Let x , xy , and xy^2 represent the numbers.

Then, $x + xy = 10$ (1)

and $xy^2 - xy = 24$ (2)

From (1), $x = \frac{10}{1+y}$ (3)

From (2), $x = \frac{24}{y^2 - y}$ (4)

Hence, $\frac{10}{1+y} = \frac{24}{y^2 - y}$ (5)

Clearing of fractions, $10y^2 - 10y = 24 + 24y$ (6)

Whence, $10y^2 - 34y = 24$ (7)

Dividing by 10, $y^2 - \frac{17}{5}y = \frac{12}{5}$ (8)

Completing square, $y^2 - \frac{17}{5}y + \frac{289}{100} = \frac{12}{5} + \frac{289}{100}$ (9)

Evolving, $y - \frac{17}{10} = \pm \frac{23}{10}$ (10)

Whence, $y = 4$, or $-\frac{3}{5}$

Substituting in (3), $x = \frac{10}{1+4}$, or $\frac{10}{1-\frac{3}{5}}$

Reducing, $x = 2$, or 25

$xy = 2 \times 4$, or $-\frac{3}{5} \times 25 = 8$, or -15

$xy^2 = 2 \times 16$, or $25 \times \frac{9}{25} = 32$, or 9

38. Let
and

$x =$ the greater number,

$y =$ the less number.

Then, $x - y = 8$ (1)

and $xy = 105$ (2)

Squaring (1), $x^2 - 2xy + y^2 = 64$ (3)

Multiplying (2) by 4, $4xy = 420$ (4)

Adding (3) and (4), $x^2 + 2xy + y^2 = 484$ (5)

Evolving, $x + y = \pm 22$ (6)

Equation (1), $x - y = 8$

Adding (6) and (1), $2x = 30$, or -14

Whence, $x = 15$, or -7

Subtracting (1) from (6), $2y = 14$, or -30

Whence, $y = 7$, or -15

NOTE. If only one unknown quantity be employed, or if we eliminate y from (1) and (2), we obtain

$$x^2 - 8x = 105, \text{ whence } x = 15, \text{ or } -7, \text{ and } x - 8 = 7, \text{ or } -15$$

39. Let $x =$ the number of days.

Then $S = \frac{1}{2}n(a + l) = \frac{1}{2}x(1 + x) = \frac{x^2 + x}{2} =$ dist. A travels,

and $20x =$ distance B travels.

Therefore, $\frac{x^2 + x}{2} + 20x = 23661$

Clearing of fractions, $x^2 + x + 40x = 47322$

Or, $x^2 + 41x = 47322$

Completing square, $x^2 + 41x + \frac{1681}{4} = \frac{190969}{4}$

Evolving, $x + \frac{41}{2} = \pm \frac{437}{2}$

Whence, $x = 198$, or -239

$$\frac{x^2 + x}{2} = \frac{39204 + 198}{2} = 19701$$

$$20x = 20 \times 198 = 3960$$

40. Let $x =$ the numerator,

and $y =$ the denominator.

Then $\frac{x}{y} =$ the fraction.

From the conditions, $\frac{x + a}{y} = b$ (1)

Also, $\frac{x}{y + c} = d$ (2)

From (1), $by - x = a$ (3)

From (2), $x - dy = cd$ (4)

Adding (3) and (4), $by - dy = a + cd$ (5)

Whence, $y = \frac{a + cd}{b - d}$ (6)

Multiplying (3) by d , $bdy - dx = ad$ (7)

Multiplying (4) by b , $bx - bdy = bcd$ (8)

Adding (7) and (8), $bx - dx = ad + bcd$ (9)

Whence, $x = \frac{ad + bcd}{b - d}$ (10)

Therefore, $\frac{x}{y} = \frac{\frac{ad + bcd}{b - d}}{\frac{a + cd}{b - d}}$ (11)

41. $x^{-\frac{2}{3}} = \frac{1}{x^{\frac{2}{3}}} = \frac{1}{\sqrt[3]{x^2}}$, Ans.

42. Let $x =$ the number of days.
 $2 \times 8 \times 32 =$ no. hours labor for one man.

Then, $abx = 2 \times 8 \times 32 = 512$

Dividing by ab , $x = \frac{512}{ab}$, Ans.

43. Let $x =$ one part,
 and $mx =$ the other.

Then, $mx + x = m$

Whence, $x = \frac{m}{m + 1}$

Multiplying by m , $mx = \frac{m^2}{m + 1}$

44. Let $x =$ A's stock,

and $416 - x =$ B's stock.

Then $228 - x =$ A's gain,

and $252 - (416 - x) = x - 164 =$ B's gain.

Each man's gain is proportional to the product of his capital by the time for which it was employed.

Hence, $228 - x : x - 164 :: 9x : 6(416 - x)$

By Art. 325, $228 - x : x - 164 :: 3x : 832 - 2x$

By Art. 315, $189696 - 1288x + 2x^2 = 3x^2 - 492x$

Whence, $x^2 + 796x = 189696$

Completing square, $x^2 + 796x + (398)^2 = 348100$

Evolving, $x + 398 = \pm 590$

Whence, $x = 192$

Then, $416 - x = 224$

45. The whole debt must be $\$30,000,000 \div .03 = \$1,000,000,000$, and the debt after paying $\$200,000,000$ will be $\$800,000,000$.

Let $x =$ reduced rate per cent.

Then, $1,000,000,000 \times \frac{x}{100} = 800,000,000 \times \frac{3}{100}$

Or, $10,000,000x = 24,000,000$

Dividing by 1,000,000, $10x = 24$

Whence, $x = 2.4 = 2\frac{2}{5}$

$$46. (2c - 3r)x - (c - 1)x - (c - 2r)x - x \\ = 2cx - 3rx - cx + x - cx + 2rx - x = -rx.$$

$$47. \frac{1}{a-b} = \frac{1}{a-b} \times \frac{ab}{ab} = \frac{ab}{ab(a-b)} \\ \frac{1}{a} - \frac{1}{b} = \frac{b}{ab} - \frac{a}{ab} = \frac{b-a}{ab} = \frac{b-a}{ab} \times \frac{a-b}{a-b} = \frac{2ab - a^2 - b^2}{ab(a-b)} \\ \frac{ab}{ab(a-b)} - \frac{2ab - a^2 - b^2}{ab(a-b)} = \frac{ab - 2ab + a^2 + b^2}{ab(a-b)} = \frac{a^2 - ab + b^2}{ab(a-b)}, \text{Ans.}$$

48. Ans. They have the same value.

49. By Art. 91, $(a - b) \div (\sqrt{a} - \sqrt{b}) = \sqrt{a} + \sqrt{b}$, Ans.

50. Let $x =$ the number of turkeys,
and $y =$ the number of geese.

From the conditions, $3x - 2y = 15$ (1)

Also, $y + 10 : x - 15 :: 7 : 3$ (2)

From (2), $3y + 30 = 7x - 105$ (3)

Or, $7x - 3y = 135$ (4)

Multiplying (4) by 2, $14x - 6y = 270$ (5)

Multiplying (1) by 3, $9x - 6y = 45$ (6)

Subtracting (6) from (5), $5x = 225$ (7)

Whence, $x = 45$ (8)

Substituting in (1), $135 - 2y = 15$ (9)

Whence, $2y = 120$ (10)

Or, $y = 60$ (11)

51. Let $x =$ no. lbs. of powder.

Then $\frac{2x}{3} + 10 =$ no. lbs. of nitre,

$\frac{x}{6} - 4\frac{1}{2} =$ no. lbs. of sulphur,

and $\frac{2x}{21} + \frac{10}{7} - 2 = \frac{2x}{21} - \frac{4}{7} =$ no. lbs. of charcoal.

Therefore, $\frac{2x}{3} + 10 + \frac{x}{6} - 4\frac{1}{2} + \frac{2x}{21} - \frac{4}{7} = x$

Mult. by 42, $28x + 420 + 7x - 189 + 4x - 24 = 42x$

Transp. and uniting, $-3x = -207$

Dividing by -3 , $x = 69$

52. Let $x =$ increase in razors, per dollar,

and $y =$ increase in knife-blades, per dollar.

Then $5x + 5y =$ whole increase in value.

Therefore, $5x + 5y + 10 = 444$ (1)

Also, $2\frac{2}{3}x + 1\frac{2}{3}y + 10 = 474$ (2)

Transp. and uniting in (1), $5x + 5y = 434$ (3)

Multiplying (2) by $\frac{3}{2}$, etc., $10x + 5y = 696$ (4)

Subtracting (3) from (4), $5x = 262$ (5)

Dividing by 5, $x = 52.40$ (6)

Substituting (5) in (3), $262 + 5y = 434$ (7)

Transp. and uniting, $5y = 172$ (8)

Dividing by 5, $y = 34.40$ (9)

53. Let $x =$ number of men,

and $y =$ number of widows.

Then, $3x + 3y = 110 + 1$ (1)

Also, $\frac{13x}{6} + \frac{7y}{2} = 110 - \frac{1}{2}$ (2)

Clearing (2) of fractions, etc., $13x + 21y = 657$ (3)

Multiplying (1) by 7, etc., $21x + 21y = 777$ (4)

Subtracting (3) from (4), $8x = 120$ (5)

Dividing by 8, $x = 15$ (6)

Dividing (1) by 3, $x + y = 37$ (7)

Substituting value of x , $15 + y = 37$ (8)

Whence, $y = 22$ (9)

54. Let $x =$ the third digit,
 $y =$ the fourth digit,
 and $x + y =$ the second digit.

Then, $\frac{1 + x + y + (x + y)}{3} = x$ (1)

Also, $\frac{1 + (x + y)}{4} = y$ (2)

From (1), $x - 2y = 1$ (3)

From (2), $x - 3y = -1$ (4)

Subtracting (4) from (3), $y = 2$ (5)

Substituting in (3), $x - 4 = 1$ (6)

Whence, $x = 5$ (7)

Adding (7) and (5), $x + y = 7$ (8)

Hence, $1752 =$ the number.

55. Let $x =$ time required by one cow,
 $2 \times 40 = 80 =$ time required by one horse.

Then, $\frac{7}{80} + \frac{4}{x} = \frac{1}{10}$

Transp. and uniting, $\frac{4}{x} = \frac{1}{80}$

Whence, $x = 320$

56. Let $x =$ number of sheep bought.

Then $\frac{94}{x} =$ cost of each,

and

$$\frac{x-7}{4} = \text{number sold.}$$

Therefore,

$$\frac{94}{x} \times \frac{x-7}{4} = 20$$

Clearing of fractions,

$$94x - 658 = 80x$$

Transposing and uniting,

$$14x = 658$$

Dividing by 14,

$$x = 47$$

57. Let

x = the number of coins.

Then $\sqrt{x-130}$ = no. in a side of first square,

and $\sqrt{x-31}$ = no. in a side of second square.

Hence,

$$\sqrt{x-130} + 3 = \sqrt{x-31}$$

Squaring, $x-130 + 6\sqrt{x-130} + 9 = x-31$

Transposing and uniting, $6\sqrt{x-130} = 90$

Dividing by 6,

$$\sqrt{x-130} = 15$$

Squaring,

$$x-130 = 225$$

Whence,

$$x = 355$$

NOTE. Radicals may be avoided thus: Let x and $x+3$ represent the number of coins on a side of each square, and $x^2 + 130 = (x+3)^2 + 31$, whence $x = 15$, and $x^2 + 130 = 355$.

58. Let

x = distance traveled.

Then $\frac{x}{5\frac{1}{4}} = \frac{4x}{21}$ = no. revolutions of fore wheel,

and $\frac{x}{7\frac{1}{8}} = \frac{8x}{57}$ = no. revolutions of hind wheel.

Therefore,

$$\frac{4x}{21} - \frac{8x}{57} = 2000$$

Multiplying by $\frac{3}{4}$,

$$\frac{x}{7} - \frac{2x}{19} = 1500$$

Clearing of fractions, $19x - 14x = 199500$

Uniting terms,

$$5x = 199500$$

Dividing by 20,

$$x = 39900$$

$$39900 \text{ ft.} = 13300 \text{ yds.} = 7\frac{1}{2} \text{ m. } 100 \text{ yds.}$$

59. Let $x =$ the thickness,
 $5x =$ the height,
 and $40x =$ the length.
 Then $x \times 5x \times 40x = 200x^3 =$ the contents.

Hence, $200x^3 = 5400$
 Dividing by 200, $x^3 = 27$
 Evolving, $x = 3$
 Whence, $5x = 15$
 Also, $40x = 120$

60. Let $2x =$ the whole number.
 Then $x^{\frac{1}{2}} =$ no. for Gulf of Mexico,
 and $\frac{7}{8} \times 2x = \frac{7x}{4} =$ no. for Pacific coast.

Therefore, $x^{\frac{1}{2}} + \frac{7x}{4} + 8 = 2x$
 Clear. of fract., $4x^{\frac{1}{2}} + 7x + 32 = 8x$
 Whence, $x - 4x^{\frac{1}{2}} = 32$
 Compl. square, $x - 4x^{\frac{1}{2}} + 4 = 36$
 Evolving, $x^{\frac{1}{2}} - 2 = \pm 6$
 Whence, $x^{\frac{1}{2}} = 8, \text{ or } -4$
 Squaring, $x = 64, \text{ or } 16$
 Multiplying by 2, $2x = 128, \text{ or } 32$

NOTE. The fractional exponent can be avoided by assuming $2x^2$ for the whole number.

61. Let $3x =$ A's quota,
 and $5x =$ B's quota.
 Then $\frac{7}{8} \times 5x = \frac{35x}{8} =$ C's quota.

Therefore, $3x + 5x + \frac{35x}{8} = 594$
 Clearing of fractions, $24x + 40x + 35x = 4752$

Uniting terms,	$99x = 4752$
Dividing by 99,	$x = 48$
Whence,	$3x = 144$
Then,	$5x = 240$
Also,	$\frac{35x}{8} = 210$

NOTE. The above problem may also be solved with either two or three unknown quantities.

62. Let	$x = \text{no. acres pasture,}$
and	$y = \text{no. acres arable land.}$
Then	$\frac{3x}{2} = \text{rent of } x \text{ acres pasture,}$
and	$3y = \text{rent of } y \text{ acres arable land.}$
Therefore,	$\frac{3x}{2} + 3y = 390 \quad (1)$
Also,	$y : \frac{y-x}{2} :: 5 : 1 \quad (2)$
From (1),	$x + 2y = 260 \quad (3)$
From (2),	$5x - 3y = 0 \quad (4)$
Multiplying (3) by 5,	$5x + 10y = 1300 \quad (5)$
Subtracting (4) from (5),	$13y = 1300 \quad (6)$
Dividing by 13,	$y = 100 \quad (7)$
Substituting in (3),	$x + 200 = 260 \quad (8)$
Whence,	$x = 60 \quad (9)$

63. Let	$x = \text{price of tea,}$
and	$y = \text{price of coffee.}$
Then,	$9x + 7y = 300 \quad (1)$
and	$6x + 13y = 300 \quad (2)$
Multiplying (1) by 2,	$18x + 14y = 600 \quad (3)$
Multiplying (2) by 3,	$18x + 39y = 900 \quad (4)$
Subtracting (3) from (4),	$25y = 300 \quad (5)$
Dividing by 25,	$y = 12 \quad (6)$

Substituting in (1), $9x + 84 = 300$ (7)

Whence, $x = 24$ (8)

64. The distances form a series in arithmetical progression, in which $n = 200$, $d = 2 \times 2 \text{ ft.} = 4 \text{ ft.}$, and $a = 20 \times 2 \times 3 \text{ ft.} = 120 \text{ ft.}$, to find S .

$$l = a + (n-1)d = 120 + 199 \times 4 = 120 + 796 = 916$$

$$S = \frac{1}{2}n(a+l) = 100 \times (120 + 916) = 103600$$

$$103600 \text{ ft.} = \frac{103600}{5280} \text{ miles} = 19\frac{1}{3} \text{ miles.}$$

65. By Art. 337, $\frac{(1+x) + (1-x)}{2} = \frac{2}{2} = 1$, Ans.

66. The successive crops form a series in geometrical progression, in which $a = 1 \text{ peck} = \frac{1}{4} \text{ bushel}$, $n = 5$, and $l = 12656\frac{1}{4}$, to find r .

Equa. (5), Art. 346, $r = \left(\frac{l}{a}\right)^{\frac{1}{n-1}} = \left(\frac{12656\frac{1}{4}}{\frac{1}{4}}\right)^{\frac{1}{4}} = (50625)^{\frac{1}{4}} = 15$.

67. $r = \left(\frac{l}{a}\right)^{\frac{1}{n-1}} = \left(\frac{2}{\frac{1}{27}}\right)^{\frac{1}{3}} = \left(\frac{27}{1}\right)^{\frac{1}{3}} = \frac{3}{2}$

$$S = \frac{rl - a}{r - 1} = \frac{\frac{3}{2} \times 2 - \frac{1}{27}}{\frac{3}{2} - 1} = \frac{3 - \frac{1}{27}}{\frac{1}{2}} = 6 - \frac{27}{2} = 4\frac{1}{2}$$

68. Let x = the number of trees.

Then $\sqrt{x - 11}$ = no. in a side of first square,

and $\sqrt{x + 24}$ = no. in a side of second square.

Hence, $\sqrt{x - 11} + 1 = \sqrt{x + 24}$

Squaring, $x - 11 + 2\sqrt{x - 11} + 1 = x + 24$

Whence, $\sqrt{x - 11} = 17$

Squaring, $x - 11 = 289$

Whence, $x = 300$

(See Prob. 57, Note.)

69. Let x represent the number of minute spaces passed over by the minute-hand after 1 o'clock, in order to

be just 1 minute space *behind* the hour-hand ; or, in other words, let x represent the distance of the minute-hand from the 12 mark at the required time. At 1 o'clock the hour-hand is 5 minute spaces from 12, and while the minute-hand moves over x spaces, the hour-hand must pass over $\frac{x}{12}$; hence, $5 + \frac{x}{12}$ will represent the distance of the hour-hand from the 12 mark at the required time.

$$\text{Then,} \quad 5 + \frac{x}{12} - x = 1$$

$$\text{Clearing of fractions,} \quad 60 + x - 12x = 12$$

$$\text{Whence,} \quad 11x = 48$$

$$\text{Or,} \quad x = 4\frac{4}{11}$$

Again, let y represent the time past 1 o'clock when the minute-hand is just 1 minute space *ahead* of the hour-hand.

$$\text{Then,} \quad y - \left(5 + \frac{y}{12}\right) = 1$$

$$\text{Clearing of fractions,} \quad 12y - 60 - y = 12$$

$$\text{Whence,} \quad 11y = 72$$

$$\text{Or,} \quad y = 6\frac{6}{11}$$

$$\begin{array}{ll} 70. \text{ Let} & x = \text{the smaller number,} \\ \text{and} & x + 1 = \text{the larger number.} \end{array}$$

$$\text{Then,} \quad (x + 1)^2 - x^2 = 15$$

$$\text{Or,} \quad x^2 + 2x + 1 - x^2 = 15$$

$$\text{Whence,} \quad 2x = 14$$

$$\text{Dividing by 2,} \quad x = 7$$

$$\text{Adding 1,} \quad x + 1 = 8$$

NOTE. If we employ two unknown quantities, then $x^2 - y^2 = 15$, and $x - y = 1$; whence, dividing one by the other, $x + y = 15$, and we then have the sum and difference given.

$$71. \text{ Let} \quad x(x + 1)(x + 2) = \text{the number.}$$

$$\text{Then,} \quad (x + 1)(x + 2) + x(x + 2) + x(x + 1) = 47$$

$$\text{Or,} \quad x^2 + 3x + 2 + x^2 + 2x + x^2 + x = 47$$

$$\text{Uniting terms,} \quad 3x^2 + 6x = 45$$

$$\begin{array}{ll}
 \text{Dividing by 3,} & x^2 + 2x = 15 \\
 \text{Completing square,} & x^2 + 2x + 1 = 16 \\
 \text{Evolving,} & x + 1 = \pm 4 \\
 \text{Whence,} & x = 3, \text{ or } -5 \\
 \text{Then, } x(x+1)(x+2) = 3 \times 4 \times 5 = 60
 \end{array}$$

$$\begin{array}{ll}
 72. \text{ Let} & x = \text{the number of years.} \\
 \text{Then,} & 39 + x = 3(x + 17) \\
 \text{Or,} & 39 + x = 3x + 51 \\
 \text{Whence,} & -2x = 12 \\
 \text{Dividing by } -2, & x = -6, \text{ Ans.}
 \end{array}$$

The negative result shows that the gentleman was three times as old as his son 6 years ago. (Art. 180.)

$$\begin{array}{ll}
 73. \text{ Let} & x = \text{rate in still water.} \\
 \text{Then} & x + 2 = \text{rate down river,} \\
 \text{and} & x - 2 = \text{rate up river.} \\
 \text{Also } 3\frac{1}{2} \div (x+2) = \frac{7}{2x+4} = \text{time req. to row down,} \\
 \text{and } 3\frac{1}{2} \div (x-2) = \frac{7}{2x-4} = \text{time req. to row back.}
 \end{array}$$

$$\begin{array}{ll}
 \text{Therefore,} & \frac{7}{2x+4} + \frac{7}{2x-4} = 1\frac{5}{3} \\
 \text{Clear. of fract., } 21(2x-4) + 21(2x+4) = 5(4x^2-16) \\
 \text{Or,} & 84x = 20x^2 - 80 \\
 \text{Whence,} & x^2 - \frac{21}{5}x = 4 \\
 \text{Completing square,} & x^2 - \frac{21}{5}x + \frac{441}{100} = \frac{441}{100} + 4 \\
 \text{Evolving,} & x - \frac{21}{10} = \pm \frac{29}{10} \\
 \text{Whence,} & x = 5, \text{ or } -\frac{1}{5}
 \end{array}$$

$$\begin{array}{ll}
 74. \text{ Let} & x = \text{the greater number,} \\
 \text{and} & y = \text{the less number.} \\
 \text{Then,} & x + y = 9(x - y) \quad (1) \\
 \text{and} & xy = 12\frac{x}{y} + x \quad (2) \\
 \text{Dividing (2) by } x, & y = \frac{12}{y} + 1 \quad (3)
 \end{array}$$

Whence, $y^2 - y = 12$ (4)

Completing square, $y^2 - y + \frac{1}{4} = \frac{49}{4}$ (5)

Evolving, $y - \frac{1}{2} = \pm \frac{7}{2}$ (6)

Whence, $y = 4$, or -3 (7)

From (1), $10y = 8x$ (8)

Or, $4x = 5y$ (9)

Substituting value of y , $4x = 20$, or -15 (10)

Dividing by 4, $x = 5$, or $-\frac{15}{4}$ (11)

75. Let $x =$ distance A has traveled,
and $y =$ distance B has traveled.
Then $x + y =$ the whole distance.

Also, $\frac{y}{4} =$ A's rate per day,

and $\frac{x}{9} =$ B's rate per day.

Also, $x \div \frac{y}{4} = \frac{4x}{y} =$ A's time,

and $y \div \frac{x}{9} = \frac{9y}{x} =$ B's time.

Therefore, $x - y = 30$ (1)

and $\frac{4x}{y} = \frac{9y}{x}$ (2)

Clearing (2) of fractions, $4x^2 = 9y^2$ (3)

Evolving, $2x = 3y$ (4)

Whence, $x = \frac{3y}{2}$ (5)

Substituting (5) in (1), $\frac{3y}{2} - y = 30$ (6)

Whence, $y = 60$ (7)

Substituting in (5), $x = \frac{3 \times 60}{2} = 90$ (8)

Hence, $x + y = 90 + 60 = 150$ (9)

76. Let $x =$ the number.

Then, $x^3 + x^2 = 9(x + 1)$

Dividing by $x + 1$, $x^2 = 9$

Evolving, $x = 3$

77. Let $x =$ original number of men.

Then $(60 - 20)x = 40x =$ no. lbs. bread left,

$x - 5 =$ no. of crew after storm,

and $\frac{40x}{(40 + 24)(x - 5)} = \frac{5x}{8x - 40} =$ reduced daily allowance.

Therefore, $\frac{5x}{8x - 40} = \frac{5}{7}$

Clearing of fractions, $35x = 40x - 200$

Whence, $5x = 200$

Dividing by 5, $x = 40$

78. Let $x =$ no. gallons from one,

and $14 - x =$ no. gal. from other.

Then $\frac{12}{12 + 18} \times x = \frac{2x}{5} =$ no. gal. wine from 1st,

and $\frac{9}{9 + 3} \times (14 - x) = \frac{3(14 - x)}{4} =$ no. gal. wine from 2d.

Therefore, $\frac{2x}{5} + \frac{3(14 - x)}{4} = 7$

Clearing of fractions, $8x + 210 - 15x = 140$

Whence, $7x = 70$

Dividing by 7, $x = 10$

Then, $14 - x = 4.$

NOTE. If two unknown quantities be employed, then either two of the three equations, $x + y = 14$, $\frac{2x}{5} + \frac{3y}{4} = 7$, and $\frac{3x}{5} + \frac{y}{4} = 7$ may be employed. It will be seen that the second represents the amount of wine and the third the amount of water, while the first represents the whole amount drawn, and is equivalent to the sum of the second and third.

79. Let $x =$ no. hours req. by the first,

and $x + 2 =$ no. hours req. by the second.

Then, $\frac{1}{x} + \frac{1}{x + 2} = \frac{1}{1\frac{1}{2}} = \frac{8}{15}$

Clearing of fractions, $15x + 30 + 15x = 8x^2 + 16x$

Whence, $8x^2 - 14x = 30$

Completing square, $x^2 - 1\frac{1}{4}x + \frac{49}{64} = \frac{289}{64}$

$$\begin{array}{ll} \text{Evolving,} & x - \frac{7}{8} = \pm \frac{17}{8} \\ \text{Whence,} & x = 3, \text{ or } -\frac{5}{4} \\ \text{Adding 2,} & x + 2 = 5, \text{ or } \frac{3}{4} \end{array}$$

NOTE. If we use x for the second and y for the first, we shall have the two equations $x - y = 2$, and $x^{-1} + y^{-1} = \frac{8}{15}$. The latter represents the fractional parts of the work done in one day.

$$\begin{array}{ll} 80. \text{ Let} & x = \text{the first number,} \\ & y = \text{the second number,} \\ \text{and} & z = \text{the third number.} \end{array}$$

$$\text{Then,} \quad \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 9 \quad (1)$$

$$\frac{2}{x} + \frac{3}{y} = 13 \quad (2)$$

$$\text{and} \quad 8x + 3y = 5 \quad (3)$$

$$\text{From (3),} \quad y = \frac{5 - 8x}{3} \quad (4)$$

$$\text{Substituting in (2),} \quad \frac{2}{x} + \frac{9}{5 - 8x} = 13 \quad (5)$$

$$\text{Clear. of fract.,} \quad 10 - 16x + 9x = 65x - 104x^2 \quad (6)$$

$$\text{Whence,} \quad 104x^2 - 72x = -10 \quad (7)$$

$$\text{Dividing by 104,} \quad x^2 - \frac{18}{13}x = -\frac{5}{52} \quad (8)$$

$$\text{Compl. square,} \quad x^2 - \frac{18}{13}x + \frac{81}{169} = \frac{16}{169} \quad (9)$$

$$\text{Evolving,} \quad x - \frac{9}{13} = \pm \frac{4}{13}$$

$$\text{Whence,} \quad x = \frac{1}{2}, \text{ or } \frac{5}{26}$$

$$\text{Substituting in (4),} \quad y = \frac{5 - 4}{3}, \text{ or } \frac{5 - \frac{19}{13}}{3}$$

$$\text{Reducing,} \quad y = \frac{1}{3}, \text{ or } \frac{1}{13}$$

$$\text{From (1),} \quad \frac{1}{z} = 9 - \left(\frac{1}{x} + \frac{1}{y} \right)$$

$$\text{Subst. values of } x \text{ and } y, \quad \frac{1}{z} = 9 - (2 + 3), \text{ or } 9 - \left(\frac{26}{5} + \frac{13}{15} \right)$$

$$\text{Or,} \quad \frac{1}{z} = 4, \text{ or } \frac{44}{15}$$

$$\text{Whence,} \quad z = \frac{1}{4}, \text{ or } \frac{15}{44}$$

81. Let $2x =$ each child's share, after paying duty,
and $x =$ each brother's share, after paying duty.

As the legacy duties are one and three per cent, the children would receive 99 per cent, and the brothers 97 per cent, of their original shares.

Then $2x \div .99 = \frac{200x}{99} =$ each child's share before paying duty.

and $x \div .97 = \frac{100x}{97} =$ each brother's share before paying duty.

Therefore, $5 \left(\frac{200x}{99} \right) + 3 \left(\frac{100x}{97} \right) = 12670$

Clear. of fractions, $97000x + 29700x = 121670010$

Or, $12670x = 12167001$

Whence, $x = 960.30$

Multiplying by 2, $2x = 1920.60$

82. Let $x =$ digit in ten's place,
and $y =$ digit in unit's place.
Then $10x + y =$ the number.

From the conditions, $10x + y = 3(x + y)$ (1)

Also, $3(10x + y) = (x + y)^2$ (2)

Dividing (2) by (1), $3 = \frac{x + y}{3}$ (3)

Or, $x + y = 9$ (4)

Multiplying by 2, $2x + 2y = 18$ (5)

From (1), $7x - 2y = 0$ (6)

Adding (5) and (6), $9x = 18$ (7)

Whence, $x = 2$ (8)

Substituting in (4), $2 + y = 9$ (9)

Or, $y = 7$ (10)

Hence, $10x + y = 27$ (11)

83. Let $12x =$ the distance each travels.

Then $x =$ number of days B travels,

and $x + 5 =$ number of days A travels.

Also, $x + 5 =$ distance A travels during last day.

We then have, $a = 1$, $l = x + 5$, $n = x + 5$, and $S = 12x$.

Hence, $12x = \frac{1}{2}(x + 5)(1 + x + 5)$

Clear. of fract., etc., $24x = (x + 5)(x + 6)$

Or, $24x = x^2 + 11x + 30$

Whence, $x^2 - 13x = -30$

Compl. square, $x^2 - 13x + \frac{169}{4} = \frac{1}{4}$

Evolving, $x - \frac{13}{2} = \pm \frac{1}{2}$

Whence, $x = 10$, or 3

Multiplying by 12, $12x = 120$, or 36

At first B travels more rapidly than A, and therefore *B overtakes A* when one has been traveling 3 days, and the other 8, each having traveled 36 miles. As A travels one mile faster each day, he soon travels more than 12 miles a day, and consequently *A overtakes B* when one has been traveling 10 days and the other 15, each having traveled 120 miles in all, or 84 miles in the 7 days since they were together before. A is now ahead, as when they first started; but he continues to travel more rapidly than B, and consequently they can never be together again, if they travel in a straight line.

84. Let $x =$ am't drawn each time.

Then $256 - x =$ am't of vinegar left, 1st time,

and $\frac{256 - x}{256} =$ am't vinegar in 1 gal., after filling.

$\frac{(256 - x)^2}{256} =$ am't of vinegar left, 2d time,

$\frac{(256 - x)^2}{(256)^2} =$ am't vinegar in 1 gal., after filling.

$\frac{(256 - x)^3}{(256)^3} =$ am't of vinegar left, 3d time,

$\frac{(256 - x)^4}{(256)^4} =$ am't of vinegar left, 4th time.

Hence, $\frac{(256 - x)^4}{(256)^3} = 81$

Clearing of fractions, $(256 - x)^4 = 81(256)^3$

Evolving,

$$256 - x = 3(4)^2 = 192$$

Whence,

$$x = 64$$

85. In the arithmetical progression there are given $a = 4$, $d = 1$, and $S = 5350$, to find n .

Equation (6), Art. 336,

$$d = \frac{2S - 2an}{n(n-1)}$$

Substituting values of letters,

$$1 = \frac{2 \times 5350 - 2 \times 4 \times n}{n(n-1)}$$

Clearing of fractions,

$$n^2 - n = 10700 - 8n$$

Or,

$$n^2 + 7n = 10700$$

Completing square, $n^2 + 7n + \frac{49}{4} = \frac{42849}{4}$

Evolving,

$$n + \frac{7}{2} = \pm \frac{297}{2}$$

Whence,

$$n = 100, \text{ or } -107$$

NOTE. If the value of n be found in terms of the given letters, it will be $n = \frac{-2a + d \pm \sqrt{(2a-d)^2 + 8dS}}{2d}$. Substituting the values of the given letters in this formula gives the same result as above.

86. Let

 $x =$ the first number, $y =$ the second number, $12 - y =$ the third number,

and

 $14 - x =$ the fourth number.

Then,

$$\frac{y}{x} = \frac{12 - y}{y} \quad (1)$$

Also,

$$(12 - y) - y = (14 - x) - (12 - y) \quad (2)$$

From (1),

$$12x - xy = y^2 \quad (3)$$

From (2),

$$2(12 - y) = y + 14 - x \quad (4)$$

Or,

$$x = 3y - 10 \quad (5)$$

Subst. (5) in (3), $36y - 120 - 3y^2 + 10y = y^2$

(6)

Or,

$$4y^2 - 46y = -120 \quad (7)$$

Dividing by 4,

$$y^2 - \frac{23}{2}y = -\frac{30}{2} \quad (8)$$

Completing square, $y^2 - \frac{23}{2}y + \frac{529}{16} = \frac{49}{16}$

(9)

Evolving,

$$y - \frac{23}{4} = \pm \frac{7}{4}$$

Whence,

$$y = \frac{15}{2}, \text{ or } 4$$

Substituting in (5),

$$x = \frac{15}{2} - 10, \text{ or } 12 - 10$$

$$\begin{array}{ll}
 \text{Reducing,} & x = 2\frac{1}{2}, \text{ or } 2 \\
 \text{Then,} & 12 - y = 12 - 1\frac{1}{2}, \text{ or } 12 - 4 \\
 \text{Reducing,} & 12 - y = \frac{9}{2}, \text{ or } 8 \\
 \text{Also,} & 14 - x = 14 - 2\frac{1}{2}, \text{ or } 14 - 2 \\
 \text{Reducing,} & 14 - x = \frac{11}{2}, \text{ or } 12
 \end{array}$$

$$\begin{array}{ll}
 87. \text{ Let} & x - y = \text{the first number,} \\
 & x = \text{the second number,} \\
 \text{and} & x + y = \text{the third number.}
 \end{array}$$

$$\text{Then, } (x - y) + x + (x + y) = 15 \quad (1)$$

$$\text{Also, } \frac{x + 4}{x - y + 1} = \frac{x + y + 19}{x + 4} \quad (2)$$

$$\text{From (1), } 3x = 15 \quad (3)$$

$$\text{Or, } x = 5 \quad (4)$$

$$\text{Substituting (4) in (2), } \frac{9}{6 - y} = \frac{24 + y}{9} \quad (5)$$

$$\text{Clearing (5) of fractions, } 81 = 144 - 18y - y^2$$

$$\text{Or, } y^2 + 18y = 63$$

$$\text{Compl. square, } y^2 + 18y + 81 = 144$$

$$\text{Evolving, } y + 9 = \pm 12$$

$$\text{Whence, } y = 3, \text{ or } -21$$

$$\text{Then, } x - y = 5 - 3 = 2$$

$$\text{Also, } x + y = 5 + 3 = 8$$

$$\begin{array}{ll}
 88. \text{ Let} & x = \text{A's number of acres,} \\
 \text{and} & y = \text{B's number of acres.}
 \end{array}$$

$$\text{Then } \frac{200}{x} = \text{A's price per acre,}$$

$$\text{and } \frac{200}{y} = \text{B's price per acre.}$$

$$\text{Therefore, } x + y = 200 \quad (1)$$

$$\text{Also, } \frac{200}{x} - \frac{200}{y} = \frac{8}{4} \quad (2)$$

$$\text{Clearing (2) of fractions, } 800y - 800x = 3xy \quad (3)$$

$$\text{From (1), } y = 200 - x \quad (4)$$

$$\text{Sub. (4) in (3), } 160000 - 800x - 800x = 600x - 3x^2 \quad (5)$$

Or, $3x^2 - 2200x = -160000$ (6)

Dividing by 3, $x^2 - 2200x = -160000$ (7)

Compl. square, $x^2 - 2200x + (1100)^2 = 730000$

Evolving, $x - 1100 = \pm 100\sqrt{73}$

Whence, $x = \frac{1100 \pm 100\sqrt{73}}{3} = \frac{1100 \pm 854.4003}{3}$

Or, $x = 651.467$, or 81.867 , nearly.

Substituting in (4), $y = 200 - 81.867 = 118.133$

$$\frac{200}{x} = \frac{200}{81.867} = 2.443$$

$$\frac{200}{y} = \frac{200}{118.133} = 1.693$$

NOTE. The greater value of x , 651.467, is rejected because it is greater than the whole number of acres, and therefore gives a negative value for

y . The exact values of the last three quantities are $y = \frac{-500 \mp 100\sqrt{73}}{3}$,

$\frac{200}{x} = \frac{6}{11 \pm \sqrt{73}} = \frac{11 \mp \sqrt{73}}{8}$, and $\frac{200}{y} = \frac{6}{-5 \mp \sqrt{73}} = \frac{5 \mp \sqrt{73}}{8}$, in

which the *lower sign* of both $\pm$ and $\mp$ must be used, in order to give positive results.

89. The first point to be settled is the meaning of the expression "B received 3s. 9d. less than he otherwise would have done." It is *not* 3s. 9d. less than he would have received had the work been completed in 5 days without C's assistance, but so much less than he would have received had A and B worked at the same rate that they actually did work, and taken more than 5 days to complete reaping the field.

Let x = the time required by B to reap the whole field,
and y = the time required by C to reap the whole field.

Then $\frac{1}{9}$, $\frac{1}{x}$, and $\frac{1}{y}$ represent the fractional parts of the whole field reaped by each in one day, and $\frac{5}{9}$, $\frac{5}{x}$, and $\frac{2}{y}$ represent the whole amount done by each.

Hence, $\frac{5}{9} + \frac{5}{x} + \frac{2}{y} = 1$ (1)

C receives $\frac{2}{y}$ of 90s., or $\frac{180}{y}$. If C had not been employed, this $\frac{180}{y}$ would have been divided between A and B in proportion to the amount of work done by each, that is, in proportion to $\frac{1}{9}$ and $\frac{1}{x}$. But B's share of C's wages would have been 3s. 9d., or $\frac{15}{4}$ s.; hence A's would have been $\frac{180}{y} - \frac{15}{4}$.

Therefore,
$$\frac{180}{y} - \frac{15}{4} : \frac{15}{4} :: \frac{1}{9} : \frac{1}{x} \quad (2)$$

Or,
$$\frac{180}{xy} - \frac{15}{4x} = \frac{5}{12} \quad (3)$$

Dividing by 5,
$$\frac{36}{xy} - \frac{3}{4x} = \frac{1}{12} \quad (4)$$

Clearing of fractions,
$$432 - 9y = xy \quad (5)$$

From (1),
$$\frac{5}{x} + \frac{2}{y} = \frac{4}{9} \quad (6)$$

Clearing of fractions,
$$45y + 18x = 4xy \quad (7)$$

Whence,
$$x = \frac{45y}{4y - 18} \quad (8)$$

Substituting (8) in (5),
$$432 - 9y = \frac{45y^2}{4y - 18} \quad (9)$$

Clearing of fract.,
$$1890y - 36y^2 - 7776 = 45y^2 \quad (10)$$

Or,
$$81y^2 - 1890y = -7776 \quad (11)$$

Dividing by 81,
$$y^2 - 23y = -96 \quad (12)$$

Completing square,
$$y^2 - 23y + \left(\frac{23}{2}\right)^2 = \frac{23^2}{4} - 96 \quad (13)$$

Evolving,
$$y - \frac{23}{2} = \pm \frac{13}{2} \quad (14)$$

Whence,
$$y = 18, \text{ or } 5\frac{1}{2}$$

Substituting in (8),
$$x = \frac{45 \times 18}{72 - 18}, \text{ or } \frac{45 \times 5\frac{1}{2}}{\frac{44}{2} - 18}$$

Reducing,
$$x = 15, \text{ or } 72$$

The roots 72 and $5\frac{1}{2}$ would answer the conditions of the problem were not C called "an *inferior* workman." They would make B an extremely poor workman, and C a better workman than A.

90. Let x = the number of workmen,
 y = no. lbs. each carried at a time,
 and n = no. loads each carried per hour.
 Then $8nxy$ = whole no. lbs. of stone.

$$\text{Therefore,} \quad 7n(x+8)(y-5) = 8nxy \quad (1)$$

$$\text{Also,} \quad 9n(x-8)(y+11) = 8nxy \quad (2)$$

$$\text{From (1),} \quad 7xy + 56y - 35x - 280 = 8xy \quad (3)$$

$$\text{From (2),} \quad 9xy - 72y + 99x - 792 = 8xy \quad (4)$$

$$\text{Adding (3) and (4),} \quad -16y + 64x - 1072 = 0 \quad (5)$$

$$\text{Dividing by } -16, \text{ etc.,} \quad y = 4x - 67 \quad (6)$$

$$\text{Subst. (6) in (3),} \quad 56(4x-67) - 35x - 280 = x(4x-67) \quad (7)$$

$$\text{Whence,} \quad 4x^2 - 256x = -4032 \quad (8)$$

$$\text{Dividing by 4,} \quad x^2 - 64x = -1008 \quad (9)$$

$$\text{Completing square,} \quad x^2 - 64x + 1024 = 16 \quad (10)$$

$$\text{Evolving,} \quad x - 32 = \pm 4 \quad (11)$$

$$\text{Whence,} \quad x = 36, \text{ or } 28$$

$$\text{Substituting in (6),} \quad y = 144 - 67, \text{ or } 112 - 67$$

$$\text{Whence,} \quad y = 77, \text{ or } 45$$

THE END.

